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ELECTROMAGNETIC AND ACOUSTIC WAVE SCATTERING BY AN ASSEMBLY OF HOMOGENEOUS OR MULTILAYERED BODIES

A T MATRIX FORMALISM FOR THE STATIONARY AND STATIC FIELD PROBLEMS

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The present paper is an introduction to and a summary of a thesis consisting of the following papers:

I. T matrix for electromagnetic scattering from an arbitrary number of scatterers and representations of E(3). (together with S. Ström) Phys. Rev. D8, 3661 (1973).

II. T matrix formulation of electromagnetic scattering from multilayered scatterers. (together with S. Ström) Phys. Rev. D10, 2670 (1974).

III. Matrix formulation of acoustic scattering from an arbitrary number of scatterers. (together with S. Ström) J. Acoust. Soc. Am. 56, 771 (1974).

IV. Matrix formulation of acoustic scattering from multilayered scatterers. (together with S. Ström) J. Acoust. Soc. Am. 57, 2 (1975).

V. Matrix formulation of static field problems involving an arbitrary number of bodies. (preprint 75-12. Institute of Theoretical Physics, Göteborg.)

The notion of the scattering matrix has been used for many years in nuclear and particle physics [1] , [2] . In the description of a collision experiment this matrix gives the connection between two asymptotic states which describe the incoming wave field and the outgoing wave field respectively. The theory has been developed along two different lines. According to one line of approach one has been trying to construct the scattering matrix starting from fundamental physical axioms, supplemented by analyticity assumptions concerning appropriate measurable variables. In this form of the theory the dynamical equations of the process are inherent in the scattering matrix. In the other approach one starts from the dynamical equations from which the scattering matrix can be computed. In particle physics one has been forced to use both methods because one does not have a complete knowledge of the dynamical equations apriori. In other branches of physics, for example electromagnetic theory, the dynamical equations have been known since long and therefore the scattering matrix for any given collision experiment can in principle be calculated. Since the scattering matrix contains all the physics of a process, its construction is not an easy task even when the dynamical equations are known. Integral equations are well suited for construction of the scattering matrix. Using this method one usually obtains recursion relations for the scattering matrix in general operator form. However, P.C. Waterman succeeded in constructing the scattering matrix for electromagnetic scattering from a homogeneous body, in a spherical wave basis. This matrix is somewhat different from the one used in particle physics because it gives the scattered field at a finite distance from the scatterer. For monocromatic waves the scattering matrix in Waterman's formalism gives a relation between the incoming and outgoing wave [3] . The incoming wave $\vec{\psi}^{i}$ is expanded in regular functions $\mathrm{Re}\,\vec{\psi}_{m}$ with constant coefficients a_{m} as $\vec{\psi}^{i} = \sum_{m} a_{m} \mathrm{Re}\,\vec{\psi}_{m}$.

The scattered wave $\vec{\psi}^{s}$ is expanded in irregular functions $\vec{\psi}_{m}$, which behave like an outgoing spherical wave for large distances, with constant coefficients f_m as $\vec{\psi}^{s} = \sum_{m} f_m \vec{\psi}_m$. The radius of convergence depends on the geometry of the configuration. The scattering matrix T is defined by $f_m = \sum_{m'} T_{mm'} a_{m'}$. In the construction of the T matrix one uses the completeness of the basis functions (or gradient or curl of the basis functions according to the character of the problem) for representation of the surface field. The completeness of different sets of basis functions is discussed in Refs. [4] and [5]. It has not yet been clarified to what extent the expansions can be used in the case of surfaces with more or less severe irregularities such as edges, corners etc. The T matrix is obtained as $T = -Q(Re,Re)Q(Out,Re)^{-1}$ where the Q's are surface integral matrices with specific combinations of regular and irregular basis functions. The precise criteria for the existence of the inverse of Q(Out,Re) have not been stated so far. The method gives remarkably good agreement when compared with other methods for far field computations as noticed in Ref. [6] in various cases and by the present author in a case with lossy dielectric ellipsoids. One merit of the method is also that one can treat scalar and vector fields also in the static limit by using the same formulas. Furthermore the scattering matrix for an assembly of bodies can be expressed in terms of the scattering matrices of the individual bodies and the translation matrices for the basis functions. It is the extension to the multiple scattering problem which constitutes the main theme of the present thesis.

Our starting point is the single scatterer T matrix formalism given by P.C. Waterman. In Refs. [3] and [7] Waterman uses the completeness of the basis functions to expand both the surface fields and the integral kernel in Poincaré-Huygen's principle. For the homogeneous bodies treated in Refs. [3] and [7] the problem is transformed to that of solving a system of infinite dimensional equations with the expansion coefficients of the incoming field as known quantities and the coefficients of the scattered field and surface fields as unknowns. Thus, by eliminating the coefficients of the surface fields the T matrix is obtained as a relation between the coefficients of the incoming and scattered field, and it gives the coefficients of the scattered field due to any incoming field. In the papers I, II and V we show that this procedure is also applicable to the case of an arbitrary number of homogeneous bodies, including the static case. The only additional requirement is that the bodies have to be separated in a specific not very strong sense. The procedure is a direct extension of that of Waterman. In our extended version one uses separate expansions of the surface fields on each body in terms of basis functions related to coordinate systems inside the different bodies. By using the translation properties of the basis functions to relate the different kinds of functions to appropriate origins one obtains the total T matrix expressed in terms of the T matrices of the individual bodies. The properties and explicit realization of the translation operators for different basis functions are given in the appendices of I, II and V. One gets a system of infinite dimensional equations of which Waterman's is the simplest one (for only one body). The various expansion coefficients can be treated as vectors ($\{a_n\} = \vec{a}$, $\{f_n\} = \vec{f}$ and so on) in an infinite dimensional space on which the different translation matrices and object related matrices (i.e. the Q matrices) are operating. In this way we obtain a system of equations involving $\vec{a}$, $\vec{f}$ and $\vec{\alpha}^i$, where the $\vec{\alpha}^i$:s are the expansion coefficients of the surface field on body number i. The number of equations exceeds the number of bodies involved by one and the coefficients in the equations are noncommuting. By inspection it can be found that the Q matrices can be rearranged to give the T matrices of the various bodies. The solution of the system of equations can be written in different ways.

In paper I a procedure is given in which the T matrices of the different bodies are treated in a symmetric way. The bodies treated so far were assumed to be homogeneous. A body which consists of several layers, each of which has constant electric and magnetic properties and which consecutively enclose each other, is called multilayered. The T matrix for several multilayered bodies can also be calculated by means of the above-mentioned method, the only difference being that the individual T matrices are more complicated. In papers III, IV and V the T matrix for a multilayered body is calculated by assuming expansions for the surface fields on the surfaces surrounding the homogeneous regions and by applying Poincaré-Huygen's principle in the different regions. The result can be obtained by a recursion relation starting with the T matrix for the innermost homogeneous region and the various Q matrices for the second innermost surface and then calculating the T matrix for the innermost two-layered object. Then one continues with the T matrix for the two-layered object and the Q matrices of the next surface and so on. The results can be generalized in such a way that any system of multilayered objects can be immersed in medias enclosed by surfaces (of course with the same restrictions as before) which in turn can be taken together with other systems and immersed together and so on. A body consisting of several non-enclosing regions can be treated as a multiple system of bodies. This introduces, however, mathematical difficulties in treating the edges, corners and points which arise in such a partition.

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Matrix formulation of static field problems involving an arbitrary number of bodies

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Abstract

In the present article we give a T matrix description for static field problems which is analogous to the T matrix description of stationary scattering. We treat an arbitrary number of bodies which either are characterized by homogeneous boundary conditions of Dirichlet's or Neuman's type or which consist of consecutively enclosing homogeneous layers with different properties. Problems with prescribed static fields or field derivatives on some of the surfaces are also treated.

I. Introduction

In Refs. [1] and [2] Waterman has given a T matrix description of acoustic and electromagnetic scattering from a single homogeneous scatterer. This formulation has been extended to an arbitrary number of multilayered scatterers for both acoustic and electromagnetic scattering Refs. [3], [4], [5] and [6].

In the present article we show that the T matrix formulation is also well suited for static field problems. Thus the static field problem for an arbitrary number of bodies consisting of several homogeneous layers consecutively enclose each other, is solved for an arbitrary source. Furthermore, by combining the results in a suitable way we obtain the T matrix for one body which contains several enclosures, which may themselves be multilayered in the above sense. The surfaces of the bodies as well as the geometrical configuration of the bodies have to satisfy certain fairly weak geometrical conditions. In the case of one multilayered body a recursion formula for the T matrix itself is obtained. In the static problem the T matrix refers to expansion in spherical solutions to Laplace's equation. The translation matrices which describe a change of origin in these solutions play a central role in the problem with several bodies. We review the properties of these translation matrices and show their relation to a singular limit of a unitary and a local representation of the two-and-three-dimensional Euclidean groups (i.e., the groups of rotations and translations in two and three-dimensional space). [3], [5].

The plan of the present article is as follows. In section II we give the basic definitions and obtain the T matrix for one multilayered object, and in section III we extend the treatment to the case of an arbitrary number of multilayered objects. In section IV a field problem with prescribed fields on an arbitrary number of surfaces and a source is solved. The extension of the results in sections II and III to some geometrically complicated bodies are also discussed here. In section V we give some numerical applications. The formulae for basis functions, translation matrices as well as some general properties of the translation operators are referred to an appendix.

II. The T-matrix for a multilayered body

We shall consider a scalar field satisfying

$$\nabla^2 \psi = 0 \tag{2.1}$$

The sources determine the primary field. When bodies on which different types of boundary conditions are prescribed are taken into account they give rise to a disturbance which we call the secondary field. The determination of the secondary field will be made by means of a transition matrix formalism which is patterned on the T matrix description of stationary scattering [1] - [6]. In this formalism the transition matrix T gives the secondary field in terms of the primary field. The disturbance is caused by the bodies only, i.e. any reaction back to the sources of the primary field is neglected.

All the bodies considered in this section are passive in the sense that the Dirichlet's or Neuman's boundary conditions are homogeneous or we have a penetrating field condition. In section IV we treat the case with active bodies by which we mean a body on the surface of which the field or its normal derivative take prescribed nonzero values. For example, by this definition we call a grounded metallic body passive while a charged metallic body in electrostatics is called active. The T matrix will refer to spherical field solutions of (2.1) in the three-dimensional case and cylindrical field solutions in the two-dimensional case (cf. Refs. [1], [5] and [6]). The two- and three-dimensional cases have identical structure and they will be developed simultaneously [1], [5], [6]. We shall now consider a body consisting of several consecutively enclosing layers as in Fig. 1. The coordinate origin is chosen inside the body and the primary field is assumed to have no sources i.e. in particular no singularities inside the body i.e. it is represented by an expansion in terms of regular fields whereas the secondary field is represented by an expansion in terms of irregular fields. This is in contradistinction to the corresponding scattering problem where the scattered field is a linear combination of both regular and irregular solutions constituting an outgoing spherical wave at large distance from the scatterer. In fact the static field problem can be obtained as a long wavelength limit of the stationary scattering problem and in this limit only the irregular solutions surivive. Of course, this can also be seen from the fact that in the static problem only the irregular functions are bounded for large distance. $\psi(\vec{r})$ is the total field which is the sum of a primary field ψ^p and a secondary field ψ^s.

$$\psi(\vec{r}) = \psi^p(\vec{r}) + \psi^s(\vec{r}) \tag{2.2}$$

We shall use the following surfaceintegral representation of the field ψ :

$$\left.\begin{matrix}\psi(\vec{r})\\ 0\end{matrix}\right\} = \psi^p(\vec{r}) + \int_{S_1} d\vec{s}\,'\cdot\left\{\psi_+(\vec{r}\,')\nabla g(|\vec{r}-\vec{r}\,'|) - \left[\nabla'\psi_+(\vec{r}\,')\right] g(|\vec{r}-\vec{r}\,'|)\right\} \text{ for } \begin{cases}\vec{r} \text{ outside } S_1\\ \vec{r} \text{ inside } S_1\end{cases} \tag{2.3}$$

ψ_+ and $\nabla\psi_+$ are the limits from the outside of the total field ψ and its gradient and $g(|\vec{r}-\vec{r}\,'|)$ is the free space Green's function. The Green's functions normalized according to $\nabla^2 g(|\vec{r}-\vec{r}\,'|) = -\delta(\vec{r}-\vec{r}\,')$ are $-(\ln|\vec{r}-\vec{r}\,'|)/2\pi$ and $1/(4\pi|\vec{r}-\vec{r}\,'|)$ in the two- and three-dimensional cases respectively. The expansion of the Green's function in a complete set of solutions of Laplace's equation is written

$$g(|\vec{r}-\vec{r}\,'|) = \sum_n \lambda(n)\, \mathrm{Ir}\,\psi_n(\vec{r}_>)\, \mathrm{Re}\,\psi_n(\vec{r}_<) \tag{2.4}$$

where n is a shortened notation of several indices.

In the three-dimensional case $\lambda(n) = \frac{1}{2n+1}$ where n on the right hand side correspond to the main index of the legendre polynom. In the two-dimensional case

$$\lambda(n) = \begin{cases} \frac{1}{2n} & n \geq 1 \\ \frac{1}{2} & n = 0 \end{cases}$$

$Ir\,\psi_n$ are the irregular and $Re\,\psi_n$ the regular solutions of (2.1) (see appendix). We shall consider a source field ψ^p given by

$$\psi^p = \sum_n a_n \, Re\,\psi_n \tag{2.5}$$

(This expansion is valid out to the sources of ψ^p.) and a secondary field given by

$$\psi^s = \sum_n f_n \, Ir\,\psi_n \tag{2.6}$$

This expansion is valid for $r > r_{max}$ where $r_{max} = \max r \in S_1$ (Fig. 1). By defining new coordinate origins O_i according to the two generic cases depicted in Fig. 2 and 3 it is possible to get expansions of the secondary field valid around any prescribed point outside S_1 (cf. Ref. [7]). Fig. 2 corresponds to an expansion in irregular functions analogus to (2.6), in a new coordinate system O_i, valid for the new radius $r_i > r^i_{max}$. Fig 3, however, corresponds to an expansion in regular functions, in a new coordinate system O_i, and is valid for $r_i < r^i_{max}$ where r^i_{max} now is the shortest distance from O_i to S_1. Of course the new expansions are calculated from (2.6) by using the translation matrices for the basis functions (cf. Appendix). The transitionmatrix T with elements $T_{nn'}$ satisfies

$$f_n = \sum_{n'} T_{nn'} \, a_{n'} \tag{2.7}$$

An explicit expression for T for the case of consecutively enclosing homogeneous layers is obtained as follows. By considering $\vec{r}$ inside an in S_1 inscribed sphere with center in origin we get, using (2.3), (2.5) and after expanding the Green's function as in (2.4) and then comparing coefficients of $\mathrm{Re}\,\psi_n(\vec{r})$.

$$a_n = -\lambda(n)\int_{S_1} d\vec{S}'\cdot\{\psi_+(\vec{r}')\nabla'\mathrm{Ir}\,\psi_n(\vec{r}') - [\nabla'\psi_+(\vec{r}')]\mathrm{Ir}\,\psi_n(\vec{r}')\} \tag{2.8}$$

Equation (2.9) is obtained by considering $\vec{r}$ outside the circumscribed sphere of S_1 with center in origin using (2.3), (2.6) and after expanding the Green's function as in (2.4) and then comparing coefficients of $\mathrm{Ir}\,\psi_n(\vec{r})$.

$$f_n = \lambda(n)\int_{S_1} d\vec{S}'\cdot\{\psi_+(\vec{r}')\nabla'\mathrm{Re}\,\psi_n(\vec{r}') - [\nabla'\psi_+(\vec{r}')]\,\mathrm{Re}\,\psi_n(\vec{r}')\} \tag{2.9}$$

Boundary conditions of interest are

i) Dirichlet's condition $\psi_+ = 0$ on S_1.

ii) Neumann's condition $\hat{n}\cdot\nabla\psi_+ = 0$ on S_1.

iii) Penetrating field condition $\varrho_0\psi_+ = \varrho_1\psi_-^1$,

$$\mathcal{H}_0\,\hat{n}\cdot\nabla\psi_+ = \mathcal{H}_1\,\hat{n}\cdot\nabla\psi_-^1 \quad \text{on } S_1.$$

ψ_-^1 is the limit from the inside of the field ψ^1 between S_1 and S_2. As is well known these three boundary conditions have applications in such areas as electrostatics, magnetostatics, heatconduction and static flow problems. We want to use a suitable complete system of functions for the expansion of the boundary values of the field and its normal derivative on S_1. The problem is then reduced to a system of equations for a_n, f_n and the expansion coefficients for the surface field [1]. The completeness of the regular wave solutions to Helmholt's equation are established in Ref. [1] for the three-dimensional case and in Ref. [8] for the two-dimensional case. The completeness of the solutions to

Laplace's equation can be found in an analogous way.

When making an expansion of the surface fields it is necessary to require that the radius $r(\theta,\varphi)$ to points on the surface is continous and singlevalued.

By expanding the field on the boundary according to case

i) $$\hat{n}\cdot\nabla\psi_+ = \sum_n \alpha_n^1\, \hat{n}\cdot\nabla \operatorname{Re}\psi_n$$

ii) $$\psi_+ = \sum_n \alpha_n^1 \operatorname{Re}\psi_n$$

iii) $$\psi_-^1 = \sum_n \{\alpha_n^1 \operatorname{Re}\psi_n + \beta_n^1 \operatorname{Ir}\psi_n\}$$

and defining

i) $$Q^{D1}_{nn'}(\mathrm{Ir},\mathrm{Re}) = \lambda(n)\int_{S_1} d\vec{S}\cdot \operatorname{Ir}\psi_n(\vec{r})\,\nabla \operatorname{Re}\psi_{n'}(\vec{r})$$

ii) $$Q^{N1}_{nn'}(\mathrm{Ir},\mathrm{Re}) = -\lambda(n)\int_{S_1} d\vec{S}\cdot [\nabla \operatorname{Ir}\psi_n(\vec{r})]\operatorname{Re}\psi_{n'}(\vec{r})$$

iii) $$Q^{P1}_{nn'}(\mathrm{Ir},\mathrm{Re}) = -\lambda(n)\int_{S_1} d\vec{S}\cdot \Big\{[\nabla \operatorname{Ir}\psi_n(\vec{r})]\frac{\varrho_1}{\varrho_0}\operatorname{Re}\psi_{n'}(\vec{r}) - \frac{\varkappa_1}{\varkappa_0}\operatorname{Ir}\psi_n(\vec{r})\,\nabla \operatorname{Re}\psi_{n'}(\vec{r})\Big\} \qquad (2.10)$$

The first argument in Q^1 tells whether the function associated with the lower left index is irregular or regular and similarly for the second argument and the lower right index. We note that

$$Q^{D1}_{nn'}(\mathrm{Ir},\mathrm{Re}) = -\frac{\lambda(n)}{\lambda(n')}\, Q^{N1}_{n'n}(\mathrm{Re},\mathrm{Ir})$$

$$Q^{P1}_{nn'}(\mathrm{Ir},\mathrm{Re}) = \frac{\varrho_1}{\varrho_0}\, Q^{N1}_{nn'}(\mathrm{Ir},\mathrm{Re}) - \frac{\varkappa_1\,\lambda(n)}{\varkappa_0\,\lambda(n')}\, Q^{N1}_{n'n}(\mathrm{Re},\mathrm{Ir})$$

For a homogeneous body characterized by penetrating field boundary condition i.e. case iii), the coefficients β_n^1 have to vanish otherwise this would imply a source in the body. Thus for homogeneous bodies (case i, ii, or iii with $\mathcal{B}_n^1 \equiv 0$) the T matrix is obtained by means of an elimination of the α_n^1:s. From (2.8) and (2.9) one gets, using a vector and matrix notation $\vec{a} \equiv \{a_n\}$ etc. the equations

$$\vec{a} = Q^1(Ir, Re)\vec{\alpha}^1 \tag{2.11}$$

$$f = -Q^1(Re, Re)\vec{\alpha}^1 \tag{2.12}$$

and thus by definition

$$T = -Q^1(Re, Re)\,Q^1(Ir, Re)^{-1} \tag{2.13}$$

For the more general case of a nonhomogeneous body with another surface S_2 inside S_1 we have to require $\mathcal{B}_n^1 \not\equiv 0$. The layer between S_1 and S_2 does not contain the origin and therefore an expansion of the field in this region contains also irr. parts consequently we must assume $\mathcal{B}_n^1 \not\equiv 0$. Instead of (2.11) and (2.12) we get

$$\vec{a} = Q^1(Ir, Re)\vec{\alpha}^1 + Q^1(Ir, Ir)\vec{\mathcal{B}}^1 \tag{2.14}$$

$$\vec{f} = -Q^1(Re, Re)\vec{\alpha}^1 - Q^1(Re, Ir)\vec{\mathcal{B}}^1 \tag{2.15}$$

Because of the presence of $\vec{\mathcal{B}}^1$ the number of equations is not yet sufficient for a determination of the relation between $\vec{a}$ and $\vec{f}$ as might be expected from the fact that the properties of the region inside S_2 have not been taken into account. To get more equations we use the integral representation for the field ψ^1 between the surfaces S_1 and S_2. This yields

$$\left.\begin{matrix}\psi^1(\vec{r})\\ 0\end{matrix}\right\} = \int_{S_1}(-\hat{n}_1)\,ds'\cdot\{\psi_-^1(\vec{r}')\nabla' g(|\vec{r}-\vec{r}'|) - g(|\vec{r}-\vec{r}'|)\nabla'\psi_-^1(\vec{r}')\} + \int_{S_2}\hat{n}_2\,ds'\cdot\{\psi_+^1(\vec{r}')\nabla' g(|\vec{r}-\vec{r}'|) - g(|\vec{r}-\vec{r}'|)\nabla'\psi_+^1(\vec{r}')\}\ \text{for}\left\{\begin{matrix}\vec{r}\text{ between } S_1 \text{ and } S_2\\ \vec{r}\text{ outside } S_1 \text{ and } S_2\end{matrix}\right. \tag{2.16}$$

Here the Green's function is the same as the one used in (2.3).

We now assume penetrating field boundary conditions also on S_2 i.e.

$$\rho_1 \psi_+^1 = \rho_2 \psi_-^2 \quad , \quad \mathcal{H}_1 \hat{n}_2 \cdot \nabla \psi_+^1 = \mathcal{H}_2 \hat{n}_2 \cdot \nabla \psi_-^2$$

(The conditions i, and ii, could also be treated in an analogous way.)

where we assume that ψ_-^2 has an expansion of the form

$$\psi_-^2 = \sum_n \alpha_n^2 \, Re\,\psi_n + \beta_n^2 \, Ir\,\psi_n \qquad (2.17$$

Here $\vec{\beta}^2 \equiv 0$ if the region inside S_2 is homogeneous, but in the same way as before, $\vec{\beta}^2 \not\equiv 0$ if S_2 contains an inhomogeneity, bounded by S_3, which itself may be multilayered.

In order to outline the general structure of the problem we consider the more general case $\vec{\beta}^2 \not\equiv 0$. By considering the case of $\vec{r}$ outside the circumscribed sphere ${}^{o}\mathcal{S}_1$ of S_1 and of $\vec{r}$ inside the inscribed sphere ${}^{i}\mathcal{S}_2$ of S_2 (both with center in 0) we obtain two equations for α_n^1, β_n^1, α_n^2 and β_n^2 as follows. The Q-matrices are defined as in (2.10). Introducing the expression of ψ_-^1 (2.17) and (2.4) into (2.16) we obtain, from a consideration of $\vec{r}$ inside ${}^{i}\mathcal{S}_2$, by comparing the coefficients of $Re\,\psi_n(\vec{r})$, the equation

$$0 = -Q^1(Ir,Re)\Big|_{eq} \vec{\alpha}^1 - Q^1(Ir,Ir)\Big|_{eq} \vec{\beta}^1 + \\ + Q^2(Ir,Re)\,\vec{\alpha}^2 + Q^2(Ir,Ir)\,\vec{\beta}^2 \qquad (2.18)$$

Similarly we obtain from a consideration of $\vec{r}$ outside ${}^{o}\mathcal{S}_1$:

$$0 = -Q^1(Re,Re)\Big|_{eq} \vec{\alpha}^1 - Q^1(Re,Ir)\Big|_{eq} \vec{\beta}^1 + \\ + Q^2(Re,Re)\,\vec{\alpha}^2 + Q^2(Re,Ir)\,\vec{\beta}^2 \qquad (2.19)$$

The notation $Q(\;,\;)\Big|_{eq}$ means that $\rho_0 = \rho_1$, and $\mathcal{H}_0 = \mathcal{H}_1$.

A direct calculation, using Gauss' theorem yields

$$Q^i(Re,Re)\Big|_{eq} = Q^i(Ir,Ir)\Big|_{eq} = 0 \qquad (2.20)$$

$$Q^{i}(Ir, Re)\Big|_{eq} = -Q^{i}(Re, Ir)\Big|_{eq} = \mathbb{1} \qquad (2.21)$$

We thus have the following system of equations.

$$\vec{a} = Q^{1}(Ir, Re)\,\vec{\alpha}^{1} + Q^{1}(Ir, Ir)\,\vec{\beta}^{1} \qquad (2.22)$$

$$\vec{f} = -Q^{1}(Re, Re)\,\vec{\alpha}^{1} - Q^{1}(Re, Ir)\,\vec{\beta}^{1} \qquad (2.23)$$

$$\vec{\alpha}^{1} = Q^{2}(Ir, Re)\,\vec{\alpha}^{2} + Q^{2}(Ir, Ir)\,\vec{\beta}^{2} \qquad (2.24)$$

$$\vec{\beta}^{1} = -Q^{2}(Re, Re)\,\vec{\alpha}^{2} - Q^{2}(Re, Ir)\,\vec{\beta}^{2} \qquad (2.25)$$

It is clear that the procedure can be continued to the last surface N enclosing a homogeneous region and thus $\vec{\beta}^{N} \equiv 0$ and then the number of equations are sufficient to get a solution. (i.e. a relation $\vec{f} = T\vec{a}$)

The last set of equations reads

$$\vec{\alpha}^{N-2} = Q^{N-1}(Ir, Re)\,\vec{\alpha}^{N-1} + Q^{N-1}(Ir, Ir)\,\vec{\beta}^{N-1}$$

$$\vec{\beta}^{N-2} = -Q^{N-1}(Re, Re)\,\vec{\alpha}^{N-1} - Q^{N-1}(Re, Ir)\,\vec{\beta}^{N-1}$$

$$\vec{\alpha}^{N-1} = Q^{N}(Ir, Re)\,\vec{\alpha}^{N}$$

$$\vec{\beta}^{N-1} = -Q^{N}(Re, Re)\,\vec{\alpha}^{N}$$

To solve the system one starts by eliminating $\vec{\alpha}^{N}$ and thus getting the transition matrix T(N) for the innermost surface. We remark that here, as before, by transition matrix we mean a relation from the coefficients of the regular functions to the coefficients of the irregular functions in the expansion of the field (of course in the same region).

$$\vec{\beta}^{N-1} = T(N)\,\vec{\alpha}^{N-1} = -Q^{N}(Re,Re)\,Q^{N}(Ir,Re)^{-1}\,\vec{\alpha}^{N-1} \tag{2.26}$$

Next eliminating $\vec{\alpha}^{N-1}$ and $\vec{\beta}^{N-1}$ and thus getting the transition matrix T(N-1) for the body which is bounded by S_{N-1} and contains the (homogeneous) enclosure S_N.

$$\vec{\beta}^{N-2} = T(N-1)\,\vec{\alpha}^{N-2} = -\left[Q^{N-1}(Re,Re) - Q^{N-1}(Re,Ir)\,T(N)\right] \times \left[Q^{N-1}(Ir,Re) + Q^{N-1}(Ir,Ir)\,T(N)\right]^{-1} \vec{\alpha}^{N-2}$$

In general defining the T matrix $T(j)$, for the layered object whose outer surface is S_j, by $\vec{\beta}^{j-1} = T(j)\,\vec{\alpha}^{j-1}$ one gets the recursion relation

$$T(j-1) = -\left[Q^{j-1}(Re,Re) + Q^{j-1}(Re,Ir)\,T(j)\right] \times \left[Q^{j-1}(Ir,Re) + Q^{j-1}(Ir,Ir)\,T(j)\right]^{-1} \tag{2.27}$$

Repeated application of (2.27) starting from (2.26) determines the T matrix T(1) for the whole multilayered body.

A comparison shows that the structure of the solution to the static field problem for a multilayered body has exactly the same structure as the corresponding stationary scattering problem and the solution of the static problem is obtained by taking the limit $k \to 0$ (where k is the wave number) in the formulas for the scattering problem [4], [6]. This fact is to be expected in view of the general properties of solutions of the Helmholt's equation [9].

III The T matrix for an arbitrary number of multilayered bodies

The T matrix for an arbitrary number of multilayered bodies can be obtained as in [4], and [6]. However the translation matrices for the basisfunctions have slightly different properties in the static and stationary scattering cases. Simply letting the wave vector go to zero in the translation matrices for the spherical wave solutions to Helmholtz' equation would cause one set of the translationmatrices as a unit matrix and the other set would have all elements infinitely great. One thus has to take the limit after multiplying by appropriate powers of the wavevectors [10]. This gives three sets of translation matrices for the different functions according to the geometrical relation between arguments and translation distance. The two-dimensional static case with its logaritmic behaviour is especially complicated. One has to study these properties explicitly to see that in fact after making appropriate definitions for basis functions and translation matrices both the two- and three-dimensional static cases can be given the same algebraic structure. We consider the configuration depicted in Fig. 4. The coordinate system 0 is chosen outside all of the S_{ij}:s. Inside all of S_{ij}, O_i are origins of new coordinate systems such that the radii $\vec{r}''_{ij}$ to S_{ij} are continuous functions of their respective spherical angles. Further restrictions on the allowed configurations will be given in due course. For the surface fields we write

$$\psi_-^{ij}(\vec{r}''_{ij}) = \sum_n \left[\alpha_n^{ij}\,\mathrm{Re}\,\psi_n(\vec{r}''_{ij}) + \beta_n^{ij}\,\mathrm{Ir}\,\psi_n(\vec{r}''_{ij})\right] \qquad (3.1)$$

The upper indices refer to body number i and surface number j $i = 1, N$ $j = 1, M_i$. For the total field ψ we have the surface integral representation:

$$\left.\begin{matrix}\psi(\vec{r})\\ 0\end{matrix}\right\} = \psi^P(\vec{r}) + \int_{\sum_i S_{i1}} d\vec{S}'\cdot\left\{\psi_+(\vec{r}')\nabla' g(|\vec{r}-\vec{r}'|) - \left[\nabla'\psi_+(\vec{r}')\right] g(|\vec{r}-\vec{r}'|)\right\} \text{ for } \begin{cases}\vec{r} \text{ outside } S_{i1} \text{ all } i\\ \vec{r} \text{ inside one of } S_{i1}\end{cases} \qquad (3.2)$$

By considering $\vec{r}$ outside a sphere with center in 0 and containing all of the S_{i1} we get, introducing (3.1) by means of the boundary conditions, and expanding the Green's function as in (2.4) and then comparing the coefficients of $Ir\,\psi_n(\vec{r})$:

$$f_n = \lambda(n) \sum_{i=1,N} \Big\{ \int_{S_{i1}} ds'\, \hat{n}_{i1} \cdot \sum_{n'} \Big[\frac{\varrho_{i1}}{\varrho_0} (\nabla' Re\,\psi_n(\vec{r}'_{i1}))\, Re\,\psi_{n'}(\vec{r}''_{i1}) - \frac{\varkappa_{i1}}{\varkappa_0} Re\,\psi_n(\vec{r}'_{i1})\, \nabla'' Re\,\psi_{n'}(\vec{r}''_{i1}) \Big] \alpha^{i1}_{n'} +$$

$$+ \Big[\frac{\varrho_{i1}}{\varrho_0} (\nabla' Re\,\psi_n(\vec{r}'_{i1}))\, Ir\,\psi_{n'}(\vec{r}''_{i1}) - \frac{\varkappa_{i1}}{\varkappa_0} Re\,\psi_n(\vec{r}'_{i1}) \times \nabla'' Ir\,\psi_{n'}(\vec{r}''_{i1}) \Big] B^{i1}_{n'} \Big\} \qquad (3.3)$$

where f_n are the expansion coefficients for ψ^s as in (2.6)
Again it is possible to extend the region in which ψ^s is given by an expansion in regular or irregular basis functions, as in section II. In (3.3) we have $\vec{r}'_{i1} = \vec{a}_i + \vec{r}''_{i1}$. A translation of the origin of the $Re\,\psi_n$ functions gives

$$Re\,\psi_n(\vec{r}'_{i1}) = Re\,\psi_n(\vec{a}_i + \vec{r}''_{i1}) = \sum_{n'} R^{(r)}_{nn'}(\vec{a}_i)\, Re\,\psi_{n'}(\vec{r}''_{i1}) \qquad (3.4)$$

After introducing $\tilde{R}^{(r)}_{nn'}(\vec{a}_i) \equiv \frac{\lambda(n)}{\lambda(n')} R^{(r)}_{nn'}(\vec{a}_i)$ it follows that (3.4) may be written

$$\vec{f} = -\sum_{i=1,N} \tilde{R}^{(r)}(\vec{a}_i) \Big[Q^{i1}(Re,Re)\, \vec{\alpha}^{\,i1} + Q^{i1}(Re,Ir)\, \vec{B}^{\,i1} \Big] \qquad (3.5)$$

where the matrices Q^{i1} are defined in complete analogy with (2.10), the integration now being over the surfaces S_{i1}.

Similarly if we consider $\vec{r}$ inside the inscribed sphere of S_{i1} (with center in O_i), we find

$$\sum_{n'} a_{n'} R^{(r)}_{n'n}(\vec{a}_i) = -\lambda(n) \sum_{n'} \Big\{ \int_{S_{i1}} ds' \hat{n}_{i1} \cdot \Big(\Big[\frac{\varrho_{i1}}{\varrho_0} \times$$

$$\times [\nabla'' Ir\psi_n(\vec{r}''_{i1})] Re\psi_{n'}(\vec{r}''_{i1}) - \frac{\varkappa_{i1}}{\varkappa_0} Ir\psi_n(\vec{r}''_{i1}) \times$$

$$\times \nabla'' Re\psi_{n'}(\vec{r}''_{i1}) \Big] A^{i1}_{n'} + \Big[\frac{\varrho_{i1}}{\varrho_0} [\nabla'' Ir\psi_n(\vec{r}''_{i1})] Ir\psi_{n'}(\vec{r}''_{i1}) -$$

$$- \frac{\varkappa_{i1}}{\varkappa_0} Ir\psi_n(\vec{r}''_{i1}) \nabla'' Ir\psi_{n'}(\vec{r}''_{i1}) \Big] B^{i1}_{n'} \Big) +$$

$$+ \sum_{j \neq i} \int_{S_{j1}} ds' \hat{n}_{j1} \cdot \Big(\Big[\frac{\varrho_{j1}}{\varrho_0} [\nabla'' Ir\psi_n(\vec{r}''_{j1} + \vec{a}_j - \vec{a}_i)] Re\psi_{n'}(\vec{r}''_{j1}) -$$

$$- \frac{\varkappa_{j1}}{\varkappa_0} Ir\psi_n(\vec{r}''_{j1} + \vec{a}_j - \vec{a}_i) \nabla'' Re\psi_{n'}(\vec{r}''_{j1}) \Big] A^{j1}_{n'} +$$

$$+ \Big[\frac{\varrho_{j1}}{\varrho_0} [\nabla'' Ir\psi_n(\vec{r}''_{j1} + \vec{a}_j - \vec{a}_i)] Ir\psi_{n'}(\vec{r}''_{j1}) -$$

$$- \frac{\varkappa_{j1}}{\varkappa_0} Ir\psi_n(\vec{r}''_{j1} + \vec{a}_j - \vec{a}_i) \nabla'' Ir\psi_{n'}(\vec{r}''_{j1}) \Big] B^{j1}_{n'} \Big) \Big\}$$

$$\text{for } i = 1, N \qquad (3.6)$$

where we have used

$$\vec{r} - \vec{r}' = \vec{r} - \vec{r}_{j1} = \vec{r}_i + \vec{a}_i - \vec{r}'_{j1} = \vec{r}_i - (\vec{r}''_{j1} + \vec{a}_j - \vec{a}_i)$$

in the Green's function in the integrals over S_{j1} (Thus we have

$r_i < |\vec{r}''_{j1} + \vec{a}_j - \vec{a}_i|$), and where a_n are the expansion coefficients for ψ^p as in (2.5). In order to be able to express also Eq. (3.6) in terms of Q^{ij} matrices we now assume (as in Ref. [5]) that the configuration of the S_{j1}:s is such that $r''_{j1} < |\vec{a}_j - \vec{a}_i|$.

When this condition is fulfilled (cf. the discussion in Ref. [5]) the change of origin of the $Ir\,\psi_n$ functions is expressed by the expansion

$$Ir\,\psi_n(\vec{r}''_{j1} + \vec{a}_j - \vec{a}_i) = \sum_{n'} \mathbb{T}_{nn'}(\vec{a}_j - \vec{a}_i)\, Re\,\psi_{n'}(\vec{r}''_{j1}) \tag{3.7}$$

It should be noted that it is sufficient to be able to find one inner point O_j inside S_{j1} for which $r''_{j1} < |\vec{a}_j - \vec{a}_i|$ is fulfilled. After introducing $\tilde{\mathbb{T}}_{nn'}(\vec{c}) = \frac{\lambda(n)}{\lambda(n')}\mathbb{T}_{nn'}(\vec{c})$ we get the equation

$$\begin{aligned} R^{(r)t}(\vec{a}_i)\vec{\alpha} &= Q^{i1}(Ir, Re)\vec{\alpha}^{i1} + Q^{i1}(Ir, Ir)\vec{\beta}^{i1} + \\ &+ \sum_{j \neq i} \tilde{\mathbb{T}}(-\vec{a}_i + \vec{a}_j)\left[Q^{j1}(Re, Re)\vec{\alpha}^{j1} + Q^{j1}(Re, Ir)\vec{\beta}^{j1}\right] \\ &\qquad \text{for} \quad i = 1, N \end{aligned} \tag{3.8}$$

where $R^{(r)t}$ denotes the transpose of $R^{(r)}$. The equations which are obtained by invoking the boundary conditions on the S_{j2} are the same as given before in section II i.e. we now get

$$\vec{\alpha}^{j1} = Q^{j2}(Ir, Re)\vec{\alpha}^{j2} + Q^{j2}(Ir, Ir)\vec{\beta}^{j2} \qquad j = 1, N \tag{3.9}$$

$$\vec{\beta}^{j1} = -Q^{j2}(Re, Re)\vec{\alpha}^{j2} - Q^{j2}(Re, Ir)\vec{\beta}^{j2} \qquad j = 1, N \tag{3.10}$$

and similarly for the remaining $\vec{\alpha}^{ji}$ and $\vec{\beta}^{ji}$ $i=3, M_j$.
Since S_{jM_j} are the innermost surfaces the Q^{jM_j} matrices can also be those corresponding to Dirichlet's or Neumann's boundary conditions. It is now convenient to introduce

$$\vec{a}^j = Q^{j1}(Ir,Re)\vec{\alpha}^{j1} + Q^{j1}(Ir,Ir)\vec{\beta}^{j1} \quad j = 1, N \tag{3.11}$$

$$\vec{f}^j = -Q^{j1}(Re,Re)\vec{\alpha}^{j1} - Q^{j1}(Re,Ir)\vec{\beta}^{j1} \quad j=1, N \tag{3.12}$$

From section II (cf. Eq. (2.22)-(2.25)) it follows that $\vec{a}^j$ and $\vec{f}^j$ are related by the total T matrix for the layered body whose outser surface is S_{j1}. This T matrix will be denoted $T(j,1)$ i.e. we have

$$\vec{f}^j = T(j,1)\,\vec{a}^j \qquad j = 1, N \tag{3.13}$$

Thus (3.5) and (3.8) can be written

$$R^{(r)T}(\vec{a}_i)\vec{a} = \vec{a}^i - \sum_{j\neq i} \tilde{\sigma}(-\vec{a}_i+\vec{a}_j)\,T(j,1)\,\vec{a}^j \quad i=1, N \tag{3.14}$$

$$\vec{f} = \sum_{j=1,N} \tilde{R}^{(r)}(\vec{a}_j)\,T(j,1)\,\vec{a}^j \tag{3.15}$$

We note that the structure of the equations of this section is completely analogous to the corresponding ones for stationary scattering [5], [6]. Thus the procedure given in [5] and [3] for the determination of the total T matrix for the configuration of the N bodies can be used also in the ent case. For example the T matrix for two multilayered bodies is given by.

$$T_{12}=\tilde{R}^{(r)}(\vec{a}_1)\,T(1,1)\left[1-\tilde{\mho}(-\vec{a}_1+\vec{a}_2)\,T(2,1)\,\tilde{\mho}(-\vec{a}_2+\vec{a}_1)\,T(1,1)\right]^{-1}\times$$

$$\times\left[1+\tilde{\mho}(-\vec{a}_1+\vec{a}_2)\,T(2,1)\,R^{(r)\tau}(\vec{a}_2-\vec{a}_1)\right]R^{(r)\tau}(\vec{a}_1)+$$

$$+\tilde{R}^{(r)}(\vec{a}_2)\,T(2,1)\left[1-\tilde{\mho}(-\vec{a}_2+\vec{a}_1)\,T(1,1)\,\tilde{\mho}(-\vec{a}_1+\vec{a}_2)\,T(2,1)\right]^{-1}\times$$

$$\times\left[1+\tilde{\mho}(-\vec{a}_2+\vec{a}_1)\,T(1,1)\,R^{(r)\tau}(\vec{a}_1-\vec{a}_2)\right]R^{(r)\tau}(\vec{a}_2) \qquad (3.16)$$

We recall that the method of Refs. [3] and [5] treats the N bodies in a completely symmetric way as is illustrated by the form of the two terms in equation (3.16)

If $T(2,1)\to 0$ then $T_{12}\to\tilde{R}^{(r)}(\vec{a}_1)\,T(1,1)\,\tilde{R}^{(r)\tau}(\vec{a}_1)$.
This is just the formula for the transformation of $T(1,1)$ referred to origin O_1 to the corresponding matrix T referred to origin O.
Using the relation

$$\mathrm{Ir}\,\psi_n(\vec{a}+\vec{r})=\sum_{n'}R^{(s)}_{nn'}(\vec{a})\,\mathrm{Ir}\,\psi_{n'}(\vec{r})$$ valid for $a<r$,

we can also transform the basis functions to a new origin. This leads to a new transformed T matrix $T=\left[R^{(s)\tau}(\vec{a}_1)\right]^{-1}T(1,1)\,R^{(r)\tau}(\vec{a}_1)$.
That these two ways lead to the same answer can be seen from the relation $\tilde{R}^{(r)}(\vec{a}_1)=\left[R^{(s)\tau}(\vec{a}_1)\right]^{-1}$, which is obtained from a combination of (A17) and (A21), and which is valid for all $\vec{a}$.
The objects considered so far have consisted of consecutively enclosing layers or systems of such objects. The procedure can be generalized to objects consisting of several homogeneous but nonenclosing parts as illustrated in Fig. 6. Systems of such more general objects can in turn be enclosed in different media and so on. This follows in the same way as for the stationary scattering case treated in Refs. [4] and [6].

IV Active and passive bodies in the presence of a primary field

In this section we will study a collection of active and passive objects (as defined in section II) in the presence of a prescribed primary field generated by far-away sources. Examples of systems with active and passive objects can again be found in electrostatics, magnetostatics, heatconduction and static flow theory. As mentioned before in electrostatics metallic objects with no net charge and dielectrics are passive, charged metallic objects are active (constant potential $\neq 0$ on the surface). In magnetostatics the magnetic field can be described by a potential and magnetized materials are active (with constant potential $\neq 0$ on the surface). In stationary heatconduction we have active objects such as objects with prescribed temperature or heatsources producing heat at a given rate. The passive objects are isolated or objects held at zero temperature. Among the stationary laminar flow theory we find the active objects as sources with a net flow out of or in through their surfaces. Here the passive objects are objects with impenetrable surfaces.

We shall study the case of an arbitrary number N of surfaces with prescribed fields (cf. Fig. 5). Assuming first that there are no other sources we will determine the field outside the smallest sphere with center in origin O (Fig. 5) circumscribing the surfaces. As before it is possible to extend the region for ψ^s as in sections II and III. On the surfaces S_i we assume that we have the prescribed fields $\psi_+^i(\vec{r}_i'') = \sum_n a_n^i \operatorname{Re}\psi_n(\vec{r}_i'')$ (4.1) This condition does not fix the normal derivatives $\hat{n}_i \cdot \nabla \psi_+^i(\vec{r})$ and therefore we assume an expansion of the form $\hat{n}_i \cdot \nabla \psi_+^i(\vec{r}_i'') = \sum_n B_n^i \, \hat{n}_i \cdot \nabla \operatorname{Re}\psi_n(\vec{r}_i'')$ (4.2)

The total field $\psi(\vec{r})$ can be expanded $\psi(\vec{r}) = \sum_n f_n \operatorname{Ir}\psi_n(\vec{r})$ (4.3)

(This expansion holds for $\vec{r}$ outside the above mentioned sphere.)

By considering $\vec{r}$ outside a sphere with center in origin containing all of S_i we get, using (4.1), (4.2) and (4.3) together with (3.2) (without source term) and after expanding the Green's function as in (2.4) and then comparing the coefficients of $Ir\psi_n(\vec{r})$:

$$f_n = \lambda(n) \sum_{i=1,N} \Big\{ \int_{S_i} ds' \hat{n}_i \cdot \sum_{n'} \Big[(\nabla' Re\psi_n(\vec{r}_i')) Re\psi_{n'}(\vec{r}_i'') \alpha^i_{n'} - Re\psi_n(\vec{r}_i') \nabla'' Re\psi_{n'}(\vec{r}_i'') B^i_{n'} \Big] \Big\} \qquad (4.4)$$

Using the translation matrices and the definition (2.10) of the Q matrices it follows that (4.4) can be written

$$\vec{f} = -\sum_{i=1,N} \tilde{R}^{(r)}(\vec{a}_i) \Big[Q^{Ni}(Re,Re) \vec{\alpha}^{\,i} + Q^{Di}(Re,Re) \vec{B}^{\,i} \Big] \qquad (4.5)$$

Similarly if we consider $\vec{r}$ inside the inscribed sphere of S_i (with center in O_i), we find

$$\begin{aligned} 0 = -\lambda(n) \sum_{n'} \Big\{ & \int_{S_i} ds' \hat{n}_i \cdot \Big[(\nabla'' Ir\psi_n(\vec{r}_i'')) Re\psi_{n'}(\vec{r}_i'') \vec{\alpha}^i_{n'} - Ir\psi_n(\vec{r}_i'') \nabla'' Re\psi_{n'}(\vec{r}_i'') \vec{B}^i_{n'} \Big] + \\ & + \sum_{j\neq i} \int_{S_j} ds' \hat{n}_j \cdot \Big[(\nabla'' Ir\psi_n(\vec{r}_j'' + \vec{a}_j - \vec{a}_i)) Re\psi_{n'}(\vec{r}_j) \vec{\alpha}^j_{n'} - \\ & - Ir\psi_n(\vec{r}_j + \vec{a}_j - \vec{a}_i) \nabla'' Re\psi_{n'}(\vec{r}_j'') \vec{B}^j_{n'} \Big] \Big\} \quad \text{for} \quad i=1,N \end{aligned} \qquad (4.6)$$

As before this can be simplified to

$$\begin{aligned} 0 = {} & Q^{Ni}(Ir,Re) \vec{\alpha}^{\,i} + Q^{Di}(Ir,Re) \vec{B}^{\,i} + \\ & + \sum_{j\neq i} \tilde{T}(-\vec{a}_i + \vec{a}_j) \Big[Q^{Nj}(Re,Re) \vec{\alpha}^{\,j} + Q^{Dj}(Re,Re) \vec{B}^{\,j} \Big] \quad \text{for} \quad i=1,N \end{aligned} \qquad (4.7)$$

In deriving these equations we have made the same geometrical asumptions which were necessary for (3.7) to hold. The problem of solving the system (4.5) and (4.7) increases rapidly with increasing N. However, it can be seen directly that the solution can always be expressed in terms of $R^{(r)}_i, \mathfrak{T}, T^{Di}, T^{Ni}$ and $Q^{Ni}(Ir,Re)\vec{\alpha}^i$ where of course $T^{Di} = -Q^{Di}(Re,Re)\,Q^{Di}(Ir,Re)^{-1}$ and $T^{Ni} = -Q^{Ni}(Re,Re)Q^{Ni}(Ir,Re)^{-1}$. Q^{Di} and Q^{Ni} are the Q matrices for surface i corresponding to the Dirichlet's and Neumann's problem respectively. If instead of the field, the derivative of the field would have been prescribed we let $\vec{\alpha}^k \to \vec{\beta}^k$ and interchange the prefixes NK and DK in the final result. For $N=1$ we get

$$\vec{f} = \tilde{R}^{(r)}(\vec{a}_1)\left[T^{N1} - T^{D1}\right]Q^{N1}(Ir,Re)\,\vec{\alpha}^1 \tag{4.8}$$

There are in principle two methods of determining $\vec{\alpha}$. The first is by operating with the functional $\int_{S_1} dS\,\psi_m(\vec{r})$ on Eq. (4.1). The second method is to operate with the functional $\lambda(m)\int_{S_1} d\vec{S}\cdot(\nabla\psi_m(\vec{r}))$ on Eq. (4.1). The second method gives

$$\alpha^1_n = -\sum_m \left[Q^{N1}(Re,Re)\right]^{-1}_{nm} \lambda(m)\int_{S_1} d\vec{S}\cdot\psi^1_+(\vec{r})\,\nabla\,Re\,\psi_m(\vec{r}) \tag{4.9}$$

Using (4.9) the factor $Q^{N1}(Ir,Re)\vec{\alpha}^1$ can be written

$$\sum_m \left[T^{N1}\right]^{-1}_{nm} \lambda(m)\int_{S_1} d\vec{S}\cdot\psi^1_+(\vec{r})\,\nabla\,Re\,\psi_m(\vec{r}).$$

Next we treat the full problem with K passive and $N-K$ active bodies plus a distant source. Let us make the definition $\vec{c}^i = Q^{Ni}(Ir,Re)\vec{\alpha}^i$ and $\vec{d}^i = Q^{Di}(Ir,Re)\vec{\beta}^i$. It is easy to see that this leads to a system of equations which we get just by "adding together" (3.15) and (4.5); (3.14) and (4.7)

$$\vec{f} = \sum_{i=1,K} \hat{R}^{(r)}(\vec{a}_i)\,T(i,1)\,\vec{a}^i + \sum_{i=K+1,N} \tilde{R}^{(r)}(\vec{a}_i)\left[T^{Ni}\vec{c}^i + T^{Di}\vec{d}^i\right] \tag{4.10}$$

$$R^{(r)t}(\vec{a}_i)\,\vec{a} = \vec{a}^i\Big|_{i\le K} - \sum_{i\ne j=1,K} \tilde{\mathfrak{T}}(-\vec{a}_i+\vec{a}_j)\,T(j,1)\,\vec{a}^j +$$

$$+\,\vec{c}^i\Big|_{i>K} + \vec{d}^i\Big|_{i>K} - \sum_{i\ne j=K+1,N} \tilde{\mathfrak{T}}(-\vec{a}_i+\vec{a}_j)\left[T^{Nj}\vec{c}^j + T^{Dj}\vec{d}^j\right]$$

$$\text{for} \quad i=1,N \tag{4.11}$$

In this system $\vec{a}^{\,i}$ and $\vec{d}^{\,i}$ are unknowns which we want to eliminate in order to express $\vec{f}$ in terms of $\vec{a}$, the coefficients of the source field, and $\vec{c}^{\,i} = Q^{Ni}(\mathrm{Ir}, \mathrm{Re})\vec{\alpha}^{\,i}$ where $\vec{\alpha}^{\,i}$ are the coefficients in the expansion of the surface field on body number i. We define new matrices and vectors of dimension N, with the indices i and j in (4.10) and (4.11) as follows: $\vec{\vec{a}}$ all components equal to $\vec{a}$, $\vec{\vec{g}}$ the first k components equal to $\vec{a}^{\,i}$ for $i \leq k$ and the other $N-k$ equal to $\vec{d}^{\,i}$ for $k < i \leq N$, $\vec{\vec{c}}$ with the first k components equal to zerovectors and the other $N-k$ components equal to $\vec{c}^{\,i}$ for $k < i \leq N$.

Thus we get

$$\vec{f} = [\tilde{R}][T]\vec{\vec{g}} + [\tilde{R}][T^N]\vec{\vec{c}} \tag{4.12}$$

$$[R^t]\vec{\vec{a}} = \{\mathbb{1} - [\tilde{G}][T]\}\vec{\vec{g}} + \{\mathbb{1} - [\tilde{G}][T^N]\}\vec{\vec{c}} \tag{4.11}$$

It is easy to get the formal solution

$$\begin{aligned} \vec{f} = {} & [\tilde{R}][T]\{\mathbb{1} - [\tilde{G}][T]\}^{-1}[R^t]\vec{\vec{a}} + \\ & + [\tilde{R}]\{\mathbb{1} - [T][\tilde{G}]\}^{-1}\{[T^N] - [T^D]\}\vec{\vec{c}} \end{aligned} \tag{4.13}$$

From (4.13) we can draw several conclusions. Firstly the problem is, as expected, a superposition of two problems: One with a primary field and all bodies passive and the other that with k passive and $N-k$ active bodies. The bodies with prescribed fields have a T matrix T^D resp T^N depending upon whether the field or the gradient of the field was prescribed. Secondly every body with prescribed field gives rise to a field as in (4.8) which is "propagated" by the $\mathbb{T}$ matrices between all N bodies in all possible combinations. Thirdly we can see that the first inverse which, as mentioned before, can be calculated by the methods given in Ref. [3] also gives us a possibility to get the second inverse directly. We remark that these results can be extended also to the stationary scattering problem for a scalar or a vector field.

V. Discussion and numerical applications

The formulas given in the previous sections represent exact solutions. However, in very few cases is it possible to solve the various integrals analytically and perform the matrix multiplications and inversions algebraically. For some simple bodies of this kind there are analytical results available with which the T matrix formalism can be compared. The T matrix for a sphere is diagonal and the elements can be calculated exactly. This of course gives the same result as the standard treatment in spherical coordinates. Van Bladel [11] calculates the field inside and outside a dielectric spheroid in a homogeneous electric field (observe a missing factor $\frac{4}{3}$ in eqs. (3.60) and (3.61)). The coefficients $\alpha_{\parallel}$ and $\alpha_{\perp}$ are proportional to two T matrixelements. This exact result can also be used for the case of homogeneous Dirichlet's and Neumann's boundary condition. In Ref. [12] the field inside a dielectric spheroid in a uniform static electric field is calculated by means of a method slightly different from the present one.

In order to get analytical results one is mostly forced to treat surfaces which are coordinate surfaces of a coordinate system in which Laplace's equation is separable. For surfaces not too much different from a coordinate surface one can apply perturbation theory similar to that in Ref. [13]. The problem with two spherical bodies can be treated analytically in bispherical coordinates [14]. Further, by applying the transformation properties of the separable solutions one can, at least in principle, treat the problem with several bodies bounded by coordinate surfaces. However, because of the very limited knowledge of the translation properties of these solutions, other than the spherical ones, one mostly has to transform to spherical solutions, make use of their translation properties, and then translate back again.

Fully numerical treatments, by discretization, of the static field problem can be found e.g. in Refs. [15], [16] and [17]. The T matrix method lies somewhere in the middle between the two extremes of a fully analytic and fully numeric solution. In order to obtain numerical results, we consider the truncated solutions. The convergence properties of these solutions depend on a complicated interplay between such parameters as the dimension of the matrices, the geometrical dimensions of the various bodies and their separations, the method of numerical integration and the number of intervals used, the specific choice of origin, the curvature of the surface etc. An in-depth study of all the various questions of the dependences of the solutions on these properties lies outside the scope of the present article. Here we shall only use the elementary stability tests consisting of an increase in the dimension and the number of intervals.

There are several other tests of the computer programs which can be performed. The expressions for the Q matrices can be tested numerically for instance against the relations (2.20) and (2.21) which we recall, are valid for very general surfaces. The T and Q matrices for spherically symmetric bodies has to be diagonal, which also is easy to check. Furthermore, the T matrix can be calculated for different choices of origin. The translation matrices can then be used to transform the T matrices to the same origin and check whether they coincide.

As an illustration of the T matrix formalism developed in this paper we will give some numerical values for T matrix elements (Fig. 7) corresponding to different rotational symmetric configurations. The z-axis is taken as the axis of rotational symmetry and the objects are symmetrically situated on it. As primary field we take the constant homogeneous vector field $\mathbb{E}^{p} = (\sin\alpha\ , 0\ , \cos\alpha\ )$. This field can be expressed by $\mathbb{E}^p = -\nabla\psi^p$ where $\psi^p = \sum_n a_n \, Re\,\psi_n$ with $a_{e11} = -\sqrt{\frac{4\pi}{3}}\sin\alpha$, $a_{e01} = -\sqrt{\frac{4\pi}{3}}\cos\alpha$. For rotational symmetric configurations this is general enough. With our choice of rotation axis the T matrices for rotational symmetric configurations are diagonal in the σ and m indices. It is sufficient to consider T matrices of the type $T_{\sigma mn,\sigma' m' n'} \equiv T(m)_{nn'}\,\delta_{\sigma\sigma'}\,\delta_{mm'}$ which involve

$$Q^{\nu}\left(\begin{Bmatrix}Ir\\Re\end{Bmatrix}\begin{Bmatrix}Ir\\Re\end{Bmatrix}\right)_{\sigma mn,\sigma' m' n'} = \frac{1}{2}\left[\frac{(2n'+1)(n-m)!\,(n'-m)!}{(2n+1)(n+m)!\,(n'+m)!}\right]^{1/2} \times$$

$$\times \int d\theta \sin\theta \left[\begin{Bmatrix}(n+1)\,r^{-n}\\ -n\,r^{n+1}\end{Bmatrix}\begin{Bmatrix}r^{-n'-1}\\ r^{n'}\end{Bmatrix} P_n^m(\cos\theta) - \begin{Bmatrix}r^{-n-1}\\ r^{n}\end{Bmatrix}\begin{Bmatrix}r^{-n'-1}\\ r^{n'}\end{Bmatrix} \times\right.$$

$$\left.\times \frac{\partial r}{\partial\theta}\,\frac{1}{\sin\theta}\left((n+1)\cos\theta\, P_n^m(\cos\theta) - (n-m+1)P_{n+1}^m(\cos\theta)\right)\right] \times$$

$$\times P_{n'}^m(\cos\theta)\left(1-\delta_{m0}\,\delta_{\sigma o}\right)\delta_{\sigma\sigma'}\,\delta_{mm'}$$

The integral is zero at the endpoints of the integration interval.

From the general expressions given in the appendix it follows that the translation matrices for translations $\pm a\hat{z}$ are

$$R^{(r)}_{\sigma mn,\sigma' m' n'}(\pm a\hat{z}) = (-1)^{m+n}(2n+1)\left[\frac{(2n)!}{(2n'+1)!\,(2(n-n'))!}\right]^{1/2} \times$$

$$\times \begin{pmatrix} n & n' & n-n' \\ m & -m & 0 \end{pmatrix} a^{n-n'}\,(\pm 1)^{n-n'}\,\delta_{\sigma\sigma'}\,\delta_{\sigma e}\,\delta_{mm'}$$

for $0 \leq n' \leq n$, otherwise zero

$$R^{(S)}_{\sigma mn,\sigma' m' n'}(\pm a\hat{z}) = (-1)^{m+n}(2n+1)\left[\frac{(2n')!}{(2n+1)!\,(2(n'-n))!}\right]^{1/2} \times$$

$$\times \begin{pmatrix} n & n' & n'-n \\ m & -m & 0 \end{pmatrix} a^{n'-n} (\pm 1)^{n'-n} \delta_{\sigma\sigma'} \delta_{\sigma e} \delta_{mm'}$$

for $0 \leq n \leq n'$, otherwise zero

$$T_{\sigma mn,\sigma' m' n'}(\pm a\hat{z}) = (-1)^{m+n}(2n+1)\left[\frac{(2(n+n')+1)!}{(2n+1)!\,(2n'+1)!}\right]^{1/2} \times$$

$$\times \begin{pmatrix} n & n' & n+n' \\ m & -m & 0 \end{pmatrix} a^{-n-n'-1} (\pm 1)^{n+n'} \delta_{\sigma\sigma'} \delta_{\sigma e} \delta_{mm'}$$

for all $n \geq 0,\ n' \geq 0$

The expansion coefficients f_n for the secondary field with the source field above are simply

$$f_{e0n} = -\sqrt{\frac{4\pi}{3}}\, T_{e0n,e01} \cos\alpha$$

$$f_{e1n} = -\sqrt{\frac{4\pi}{3}}\, T_{e1n,e11} \sin\alpha$$

First we treat two prolate spheroids with semiaxes a = 0.4 and b = 0.25 with Dirichlet's and respectively Neumann's homogeneous boundrary condition case A resp. case B in Fig. 7. As the second case, called case C in Fig. 7, we treat a permeable prolate spheroid with semiaxes a = 0.8 and b = 0.5 for $\varrho_1 = \varrho_2 = \varkappa_1 = 1$, $\varkappa_2 = 2$. Thirdly we treat the above permeable spheroid now containing a spheroid with Dirichlet's boundary condition. This case is called case D in Fig. 7. Finally we treat the two spheroids in case A and B. The spheroid with Neumann's boundary condition is situated at a distance c=0.5 from origin on the positive z-axis and the spheroid with Dirichlet's boundary condition is situated at a distance c from origin on the negative z-axis.

This is called case E in Fig. 7. As pointed out before these T matrix elements are sufficient for the calculation of the secondary field due to any scalar field ϕ which correspond to a homogeneous vector field $\mathbb{F}$ given by $\mathbb{F} = -\nabla\phi$. We note the following features of the convergence properties. The T matrix elements appear to decrease slowly but the basis functions decrease very fast, compensating for this, at least for large distances. However, the convergence of the field expansion near the radius of convergence (i.e. the circumscribing sphere) was very slow. The matrix Q (Ir, Re) is unbounded as one of the indices increases which has the effect that the numerical accuracy in T might even be diminished, unless the numerical accuracy in $Q(Re, Re)$ and $Q(Ir, Re)$ is increased, when the dimension is increased. The matris $Q(Ir, Ir)$ which is used when the bodies are layered, is unbounded in two indices. This unboundednes will be compensated by a multiplication by an inner T matrix but here the requirement of better numerical accuracy when the dimension is increased is more pronounced. By the same programs we could equally well have made more succesive inclusions for surfaces other than spheroids. The two body configuration could also consist of more complicated bodies of this kind.

Acknowledgements

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Appendix:

Translation matrices for spherical solutions to the scalar Laplace equation in two and three dimensions

The basis functions and their translation properties play an essential role in our treatment of the static many body problem and therefore we shall give some relations which are useful in this context. The two and three dimensional cases are structurally the same. All relations can be derived from Ref. [5] by a limit process [10] (cf. also Refs. [18] and [19]). We first state the results for the three-dimensional case [18], [19] . Consider two linearly independent solutions to

$$\nabla^2 \psi = 0 \tag{A 1}$$

which we choose as

$$Re\,\psi_n(\vec{r}) \equiv Re\,\psi_{\left\{ {e \atop o} \right\} mn}(\vec{r}) = \gamma_{mn}^{1/2}\, r^n P_n^m(\cos\theta) \left\{ {\cos m\varphi \atop \sin m\varphi} \right\} \tag{A 2}$$

$$Ir\,\psi_n(\vec{r}) \equiv Ir\,\psi_{\left\{ {e \atop o} \right\} mn}(\vec{r}) = \gamma_{mn}^{1/2}\, r^{-n-1} P_n^m(\cos\theta) \left\{ {\cos m\varphi \atop \sin m\varphi} \right\} \tag{A 3}$$

where

$$\gamma_{mn} = \frac{\varepsilon_m (2n+1)(n-m)!}{4\pi\, (n+m)!} \qquad \varepsilon_0 = 1,\ \varepsilon_m = 2 \ \text{for } m \neq 0 \tag{A 4}$$

Let (r,θ,φ) , (a,η,ψ) and (R,Θ,ϕ) be the spherical coordinates of $\vec{r}$, $\vec{a}$ and $\vec{R}$, respectively, where $\vec{r} = \vec{a} + \vec{R}$.

By multiplying Eq. (A 1), (A 3) and (A 4) of Ref. [5] with appropriate factors of k (the wavevector k) and then taking the limit $k \to 0$ we get

$$Re\,\psi_n(\vec{r}) = \sum_{n'} R_{nn'}^{(r)}(\vec{a})\, Re\,\psi_{n'}(\vec{R}) \quad \text{all } a \text{ and } R \tag{A 5}$$

$$Ir\ \psi_n(\vec{r}) = \sum_{n'} \sigma_{nn'}(\vec{a})\ Re\ \psi_{n'}(\vec{R}) \quad for\ a > R \tag{A 6}$$

$$Ir\ \psi_n(\vec{r}) = \sum_{n'} R^{(S)}_{nn'}(\vec{a})\ Ir\ \psi_{n'}(\vec{R}) \quad for\ a < R \tag{A 7}$$

and

$$\mathcal{R}_{\sigma mn,\sigma m'n'}(\alpha,\vec{a}) = (-1)^m \frac{\sqrt{\varepsilon_m \varepsilon_{m'}}}{2}\Big[(-1)^{m'} T^{\alpha}_{mn,m'n'}(a,\eta) \times$$

$$\times \cos(m-m')\psi + (-1)^{\sigma} T^{\alpha}_{mn,-m'n'}(a,\eta) \cos(m+m')\psi\Big]$$

$$\mathcal{R}_{\sigma mn,\sigma' m'n'}(\alpha,\vec{a}) = (-1)^m \frac{\sqrt{\varepsilon_m \varepsilon_{m'}}}{2}\Big[(-1)^{\sigma'+m'} T^{\alpha}_{mn,m'n'}(a,\eta) \times$$

$$\times \sin(m-m')\psi + T^{\alpha}_{mn,-m'n'} \sin(m+m')\psi\Big], \quad \sigma \neq \sigma' \tag{A 8}$$

with $\mathcal{R}(r,\vec{a}) = R^{(r)}(\vec{a})$, $\mathcal{R}(\sigma,\vec{a}) = \sigma(\vec{a})$, $\mathcal{R}(s,\vec{a}) = R^{(S)}(\vec{a})$ (A 9)

$$T^{r}_{mn,m'n'}(a,\eta) = (-1)^{n-m'}\left[(2n+1)\frac{(2n+1)!\,(n-n'-m+m')!}{(2n'+1)!\,(2(n-n'))!\,(n-n'+m-m')!}\right]^{1/2} \times$$

$$\times \begin{pmatrix} n & n' & n-n' \\ m & -m' & m'-m \end{pmatrix} a^{n-n'} P^{m-m'}_{n-n'}(\cos\eta)$$

for $0 \leq n' \leq n$, otherwise zero (A10)

$$T^{\sigma}_{mn,m'n'}(a,\eta) = (-1)^{n+m'}(2n+1)\left[\frac{(2(n+n')+1)!\,(n+n'-m+m')!}{(2n+1)!\,(2n'+1)!\,(n+n'+m-m')!}\right]^{1/2} \times$$

$$\times \begin{pmatrix} n & n' & n+n' \\ m & -m' & m'-m \end{pmatrix} a^{-n-n'-1} P^{m-m'}_{n+n'}(\cos\eta)$$

for all $n \geq 0$, $n' \geq 0$ (A11)

$$T^{s}_{mn,m'n'}(a,\eta)=(-1)^{n+m'}\left[\frac{(2n+1)(2n')!\,(n'-n-m+m')!}{(2n)!\,(2(n'-n))!\,(n'-n+m-m')!}\right]^{1/2}\times$$

$$\times\begin{pmatrix} n & n' & n'-n \\ m & -m' & m'-m \end{pmatrix} a^{n'-n}\, P^{m-m'}_{n'-n}(\cos\eta)$$

for $0\leq n\leq n'$, otherwise zero (A12)

Here $\begin{pmatrix} i & j & k \\ l & m & n \end{pmatrix}$ is the usual 3-j symbol [20] .

By repeated application of (A 5)-(A 7) we get (cf. Eq. (A 5), (A 6) and (A 9) of Ref. [5]).

$$R^{(r)}(\vec{a})R^{(r)}(\vec{b})=R^{(r)}(\vec{c}) \qquad \text{all } a \text{ and } b \tag{A13}$$

$$R^{(s)}(\vec{a})R^{(s)}(\vec{b})=R^{(s)}(\vec{c}) \qquad \text{all } a \text{ and } b \tag{A14}$$

$$R^{(s)}(\vec{a})\,\sigma(\vec{b})=\sigma(\vec{c})=\sigma(\vec{b})\,R^{(r)}(\vec{a}) \qquad \text{for } a<b \tag{A15}$$

where $\vec{c}=\vec{a}+\vec{b}$

From (A13) and (A14) we get

$$R^{(r)}(-\vec{a})=R^{(r)}(\vec{a})^{-1} \qquad \text{all } a \tag{A16}$$

$$R^{(s)}(-\vec{a})=R^{(s)}(\vec{a})^{-1} \qquad \text{all } a \tag{A17}$$

The explicit expressions for the $\mathcal{R}(\alpha,\vec{a})$:s give

$$R^{(r)}_{\sigma mn,\sigma'm'n'}(-\vec{a})=(-1)^{n-n'}R^{(r)}_{\sigma mn,\sigma'm'n'}(\vec{a}) \quad \text{all } a \tag{A18}$$

$$R^{(s)}_{\sigma mn,\sigma'm'n'}(-\vec{a})=(-1)^{n-n'}R^{(s)}_{\sigma mn,\sigma'm'n'}(\vec{a}) \quad \text{all } a \tag{A19}$$

$$\sigma_{\sigma mn,\sigma'm'n'}(-\vec{a})=(-1)^{n+n'}\sigma_{\sigma mn,\sigma'm'n'}(\vec{a}) \quad \text{for } a>0 \tag{A20}$$

and the relation needed for consistency (cf. section III) becomes

$$R^{(s)}_{\sigma' m' n', \sigma m n}(\vec{a}) = \frac{\lambda(n)}{\lambda(n')} R^{(r)}_{\sigma m n, \sigma' m' n'}(-\vec{a}) \quad \text{all } a \tag{A21}$$

where as before $\lambda(n) = \dfrac{1}{2n+1}$

Finally, we have in the same way

$$\mathcal{T}_{\sigma' m' n', \sigma m n}(\vec{a}) = \frac{\lambda(n)}{\lambda(n')} \mathcal{T}_{\sigma m n, \sigma' m' n'}(-\vec{a}) \quad \text{for } a>0 \tag{A22}$$

Consider next the two-dimensional case and consider the solutions to (A 1), which we choose as

$$Re\,\psi_n(\vec{r}) \equiv Re\,\psi_{\left\{ e \atop o \right\} n}(\vec{r}) = \sqrt{\frac{\varepsilon_n}{2\pi}}\, r^n \left\{ \cos n\varphi \atop \sin n\varphi \right\} \quad \text{all } n \tag{A23}$$

$$Ir\,\psi_n(\vec{r}) \equiv Ir\,\psi_{\left\{ e \atop o \right\} n}(\vec{r}) = \begin{cases} \sqrt{\dfrac{\varepsilon_n}{2\pi}}\, r^{-n-1} \left\{ \cos n\varphi \atop \sin n\varphi \right\} & \text{for } n>0 \\ -\dfrac{2}{\sqrt{2\pi}} \ln r\, \delta_{\sigma e} & \text{for } n=0 \end{cases} \tag{A24}$$

Let (r,φ), (a,ψ) and (R,ϕ) be the polar coordinates of $\vec{r}$, $\vec{a}$ and $\vec{R}$, respectively, and let $\vec{r} = \vec{a} + \vec{R}$. By multiplying Eq. (A11), (A13) and (A14) of Ref. [5] with appropriate factors of k and then taking the limit $k \to 0$ we get

$$Re\,\psi_n(\vec{r}) = \sum_{n'} R^{(r)}_{nn'}(\vec{a})\, Re\,\psi_{n'}(\vec{R}) \quad \text{all } a \text{ and } R \tag{A25}$$

$$Ir\,\psi_n(\vec{r}) = \sum_{n'} \mathcal{T}_{nn'}(\vec{a})\, Re\,\psi_{n'}(\vec{R}) \quad \text{for } a>R \tag{A26}$$

$$Ir\,\psi_n(\vec{r}) = \sum_{n'} R^{(s)}_{nn'}(\vec{a})\, Ir\,\psi_{n'}(\vec{R}) \quad \text{for } a<R \tag{A27}$$

with

$$R^{(r)}_{\sigma n,\sigma n'}(\vec{a}) = \sqrt{\frac{\varepsilon_n}{\varepsilon_{n'}}}\ \frac{n!\,a^{n-n'}}{n'!\,(n-n')!}\ \cos(n-n')\psi\,(1-\delta_{\sigma 0}\delta_{n'0})$$

$$R^{(r)}_{\sigma n,\sigma' n'}(\vec{a}) = \sqrt{\frac{\varepsilon_n}{\varepsilon_{n'}}}\ \frac{n!\,a^{n-n'}(-1)^{\sigma'}}{n'!\,(n-n')!}\ \sin(n-n')\psi\,(1-\delta_{\sigma' 0}\delta_{n'0}),$$

$\sigma \neq \sigma'$

for $0 \leq n' \leq n$, otherwise zero. (A28)

$$T_{\sigma 0,\sigma' 0}(\vec{a}) = -2\ln a\ \delta_{\sigma\sigma'}\,(1-\delta_{\sigma 0})$$

and for $n \neq 0$ or $n' \neq 0$

$$T_{\sigma n,\sigma n'}(\vec{a}) = \frac{2}{\sqrt{\varepsilon_n \varepsilon_{n'}}}\ (-1)^{n'+\sigma}\ \frac{(n+n'-1)!}{(n-1)!\,n'!}\ a^{-n-n'}\ \times$$

$$\times \cos(n+n')\psi\,(1-\delta_{\sigma 0}\delta_{n0})(1-\delta_{\sigma 0}\delta_{n'0})$$

$$T_{\sigma n,\sigma' n'}(\vec{a}) = \frac{2}{\sqrt{\varepsilon_n \varepsilon_{n'}}}\ (-1)^{n'}\ \frac{(n+n'-1)!}{(n-1)!\,n'!}\ a^{-n-n'}\ \times$$

$$\times \sin(n+n')\psi\,(1-\delta_{\sigma' e}\delta_{n0})(1-\delta_{\sigma' 0}\delta_{n'0}),\ \sigma \neq \sigma'$$

for all $n \geq 0$, $n' \geq 0$ and by definition $(-1)! \equiv 1$ (A29)

$$R^{(s)}_{\sigma n,\sigma n'}(\vec{a}) = \sqrt{\frac{\varepsilon_{n'}}{\varepsilon_n}}\ (-1)^{n'-n}\ \frac{(n'-1)!}{(n-1)!\,(n'-n)!}\ a^{n'-n}\ \times$$

$$\times \cos(n'-n)\psi\,(1-\delta_{\sigma 0}\delta_{n0})$$

$$R^{(s)}_{\sigma n,\sigma' n'}(\vec{a}) = \sqrt{\frac{\varepsilon_{n'}}{\varepsilon_n}}\ (-1)^{n'-n+\sigma}\ \frac{(n'-1)!}{(n-1)!\,(n'-n)!}\ a^{n'-n}\ \times$$

$$\times \sin(n'-n)\psi\,(1-\delta_{\sigma' e}\delta_{n0}),\ \ \sigma \neq \sigma'$$

for $0 \leq n \leq n'$, otherwise zero and by definition $(-1)! \equiv 1$. (A30)

By repeated application of (A25)-(A27) we get

$$R^{(r)}(\vec{a})R^{(r)}(\vec{b}) = R^{(r)}(\vec{c}) \qquad \text{all } a \text{ and } b \tag{A31}$$

$$R^{(s)}(\vec{a})R^{(s)}(\vec{b}) = R^{(s)}(\vec{c}) \qquad \text{all } a \text{ and } b \tag{A32}$$

$$R^{(s)}(\vec{a})\mathbb{T}(\vec{b}) = \mathbb{T}(\vec{c}) = \mathbb{T}(\vec{b})R^{(r)}(\vec{a}) \qquad \text{for } a<b \tag{A33}$$

where $\vec{c} = \vec{a} + \vec{b}$

From (A31) and (A32) we get

$$R^{(r)}(-\vec{a}) = R^{(r)}(\vec{a})^{-1} \qquad \text{all } a \tag{A34}$$

$$R^{(s)}(-\vec{a}) = R^{(s)}(\vec{a})^{-1} \qquad \text{all } a \tag{A35}$$

The explicit expressions for the translation matrices give

$$R^{(r)}_{\sigma n,\sigma' n'}(-\vec{a}) = (-1)^{n-n'} R^{(r)}_{\sigma n,\sigma' n'}(\vec{a}) \qquad \text{all } a \tag{A36}$$

$$R^{(s)}_{\sigma n,\sigma' n'}(-\vec{a}) = (-1)^{n-n'} R^{(s)}_{\sigma n,\sigma' n'}(\vec{a}) \qquad \text{all } a \tag{A37}$$

$$\mathbb{T}_{\sigma n,\sigma' n'}(-\vec{a}) = (-1)^{n+n'} \mathbb{T}_{\sigma n,\sigma' n'}(\vec{a}) \qquad \text{for } a>0 \tag{A38}$$

and for consistency

$$R^{(s)}_{\sigma' n',\sigma n}(\vec{a}) = \frac{\lambda(n)}{\lambda(n')} R^{(r)}_{\sigma n,\sigma' n'}(-\vec{a}) \qquad \text{all } a \tag{A39}$$

where as before $\lambda(n) = \begin{cases} 1/2n & n>0 \\ 1/2 & n=0 \end{cases}$

Finally

$$\mathbb{T}_{\sigma' n',\sigma n}(\vec{a}) = \frac{\lambda(n)}{\lambda(n')} \mathbb{T}_{\sigma n,\sigma' n'}(-\vec{a}) \qquad \text{for } a>0 \tag{A40}$$

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Figure captions

Fig. 1 Geometry and notations for a twolayered body.

Fig. 2 Outside the spheres are different expansions of ψ^s in irregula functions valid. By successive transformations it is possible to obtain (different) expansions valid everywhere outside the convex part of S_1.

Fig. 3 Outside the sphere about 0 is an expansion of ψ^s in irregular functions valid. Inside the sphere about 0_i is an expansion of ψ^s in regular functions valid. By successive transformations it is possible to obtain (different) expansions valid everywhere outside the concave part of S_1.

Fig. 4 Geometry and notations for two twolayered bodies.

Fig. 5 Geometry and notations for two active bodies.

Fig. 6 Geometries for which the T matrix formalism is applicable.

Fig. 7 Table of T matrix elements for different rotational symmetric configurations where the z-axis is rotation axis.

Fig 1

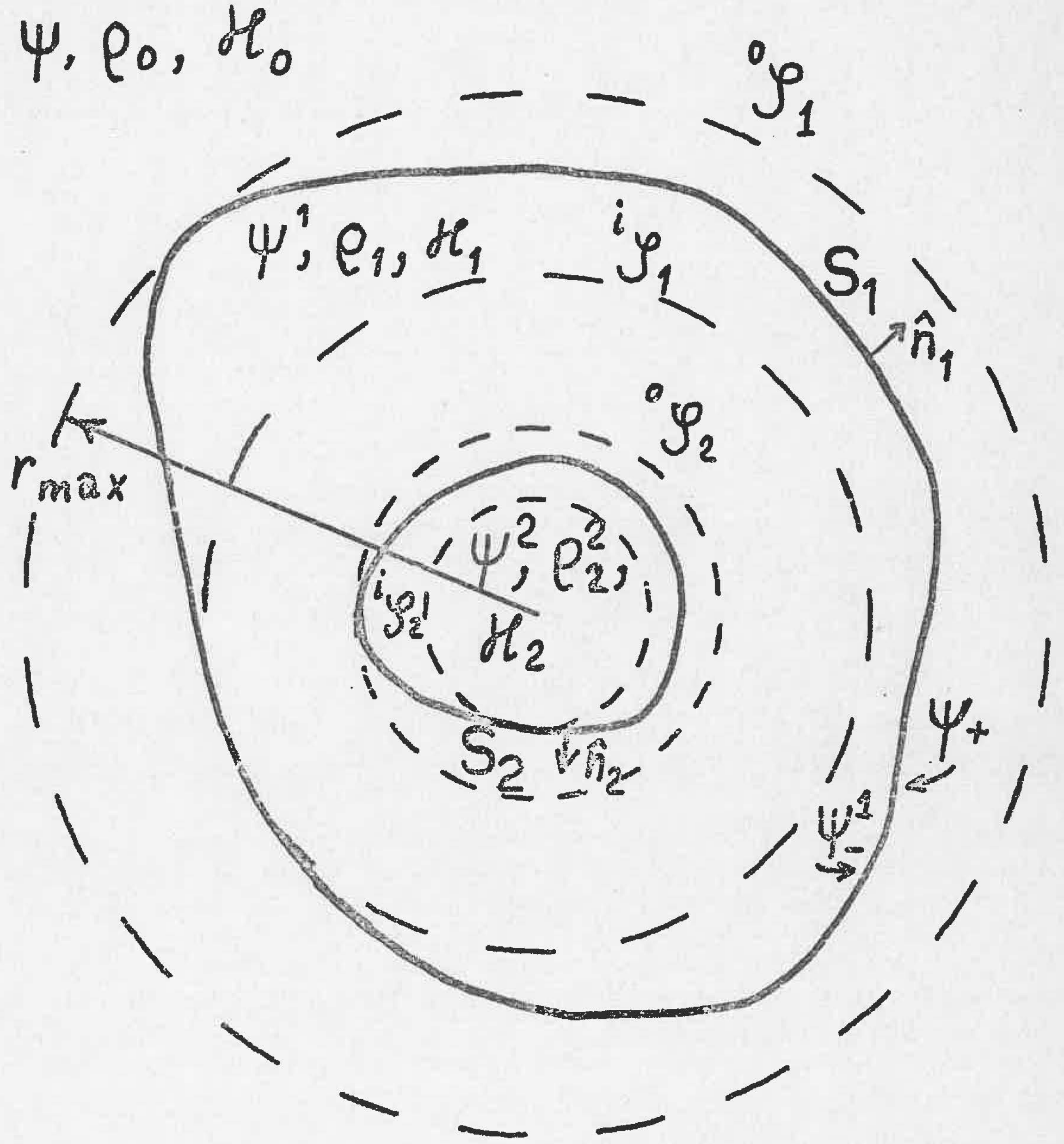

$\Psi, \rho_0, \mathcal{H}_0$
$^o\mathcal{S}_1$
$\Psi^1, \rho_1, \mathcal{H}_1$
$^i\mathcal{S}_1$
S_1
$\hat{n}_1$
$^o\mathcal{S}_2$
r_{max}
$\Psi^2, \rho_2^2,$
$^i\mathcal{S}_2$
$\mathcal{H}_2$
S_2
$\hat{n}_2$
Ψ_+
Ψ_-^1

Fig 2

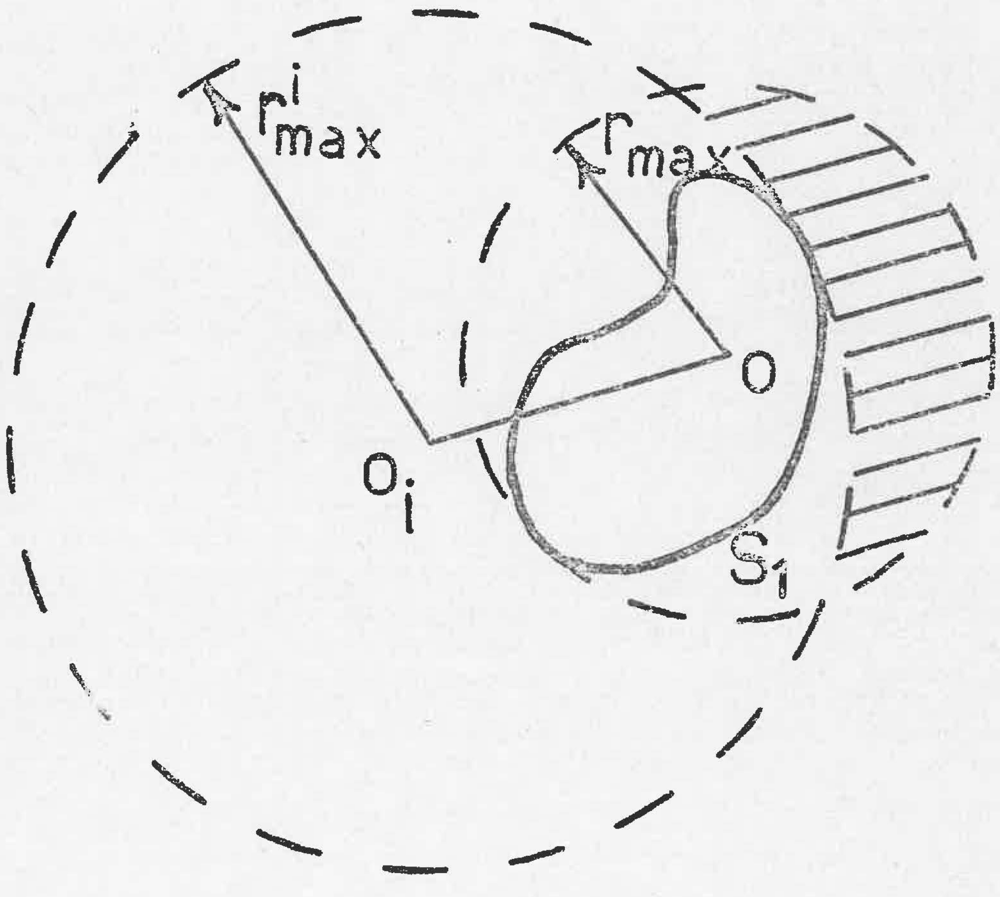


Fig 3

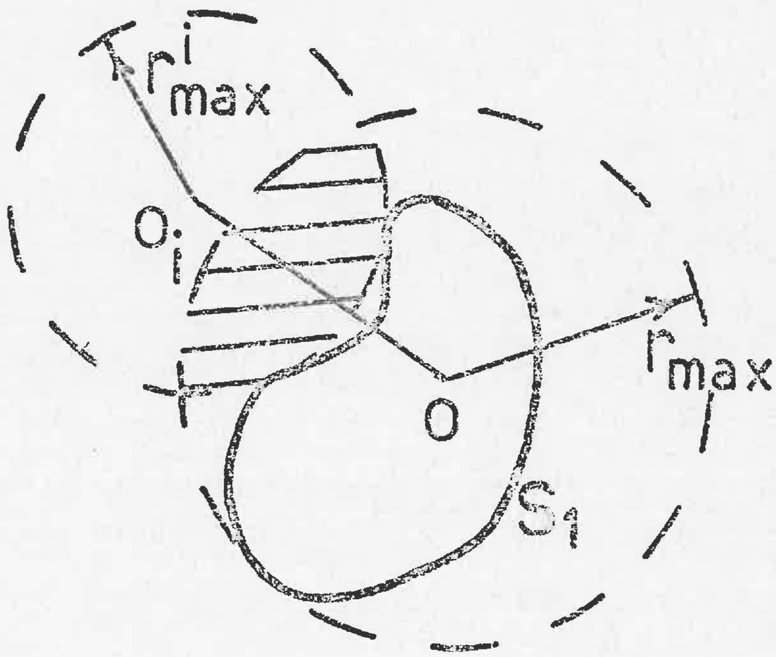
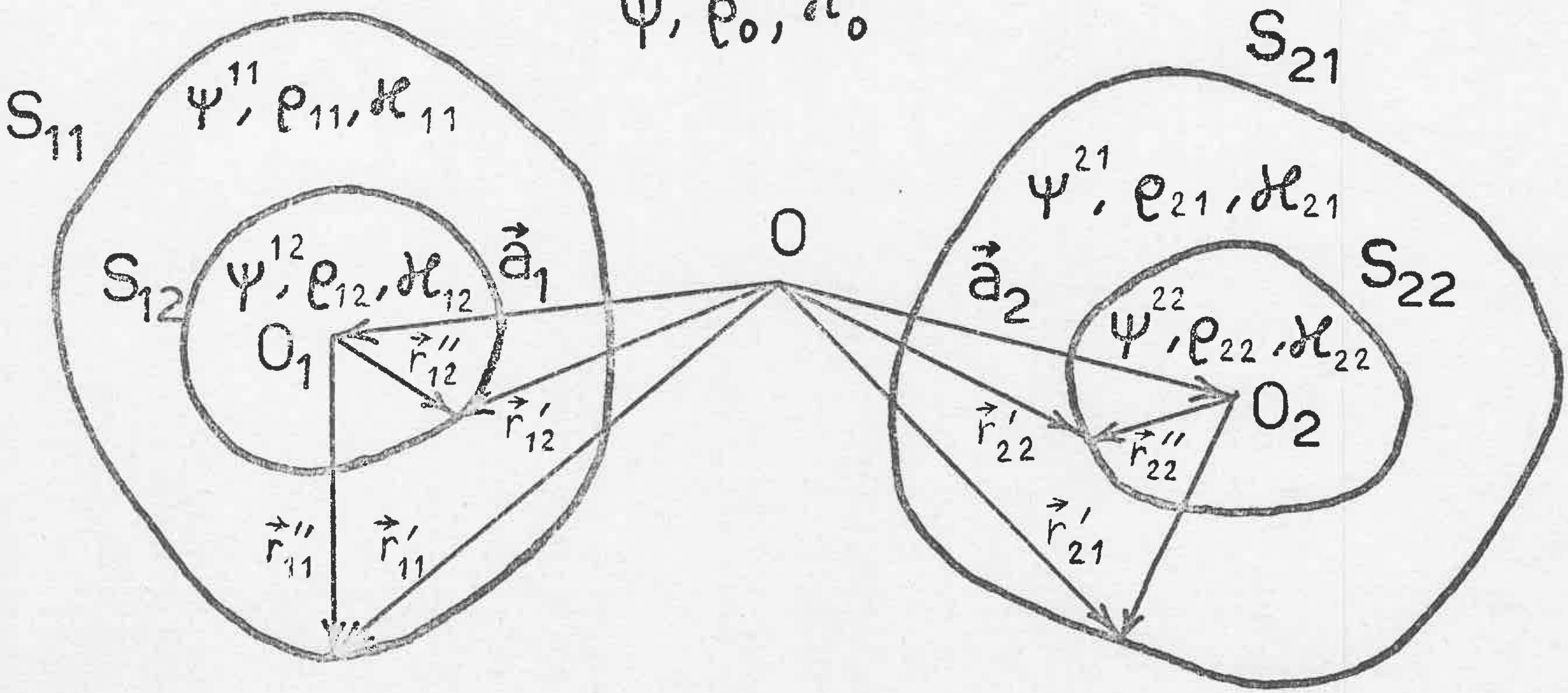


Fig 4



$\Psi, \varrho_0, \mathcal{H}_0$
S_{11}
$\Psi^{11}, \varrho_{11}, \mathcal{H}_{11}$
S_{12}
$\Psi^{12}, \varrho_{12}, \mathcal{H}_{12}$
$\vec{a}_1$
O_1
$\vec{r}\,''_{12}$
$\vec{r}\,'_{12}$
$\vec{r}\,''_{11}$
$\vec{r}\,'_{11}$
O
S_{21}
$\Psi^{21}, \varrho_{21}, \mathcal{H}_{21}$
S_{22}
$\vec{a}_2$
$\Psi^{22}, \varrho_{22}, \mathcal{H}_{22}$
O_2
$\vec{r}\,'_{22}$
$\vec{r}\,''_{22}$
$\vec{r}\,'_{21}$

Fig 5

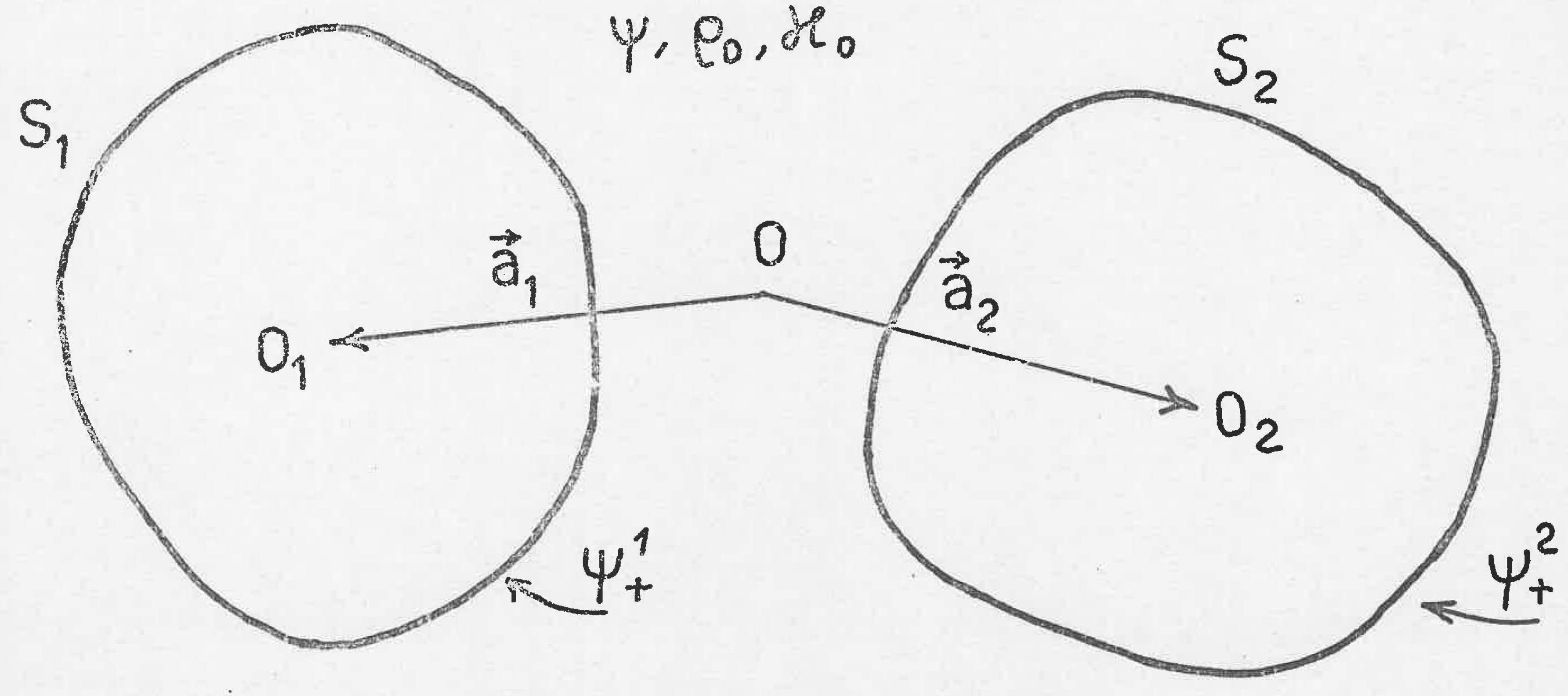


Fig 6

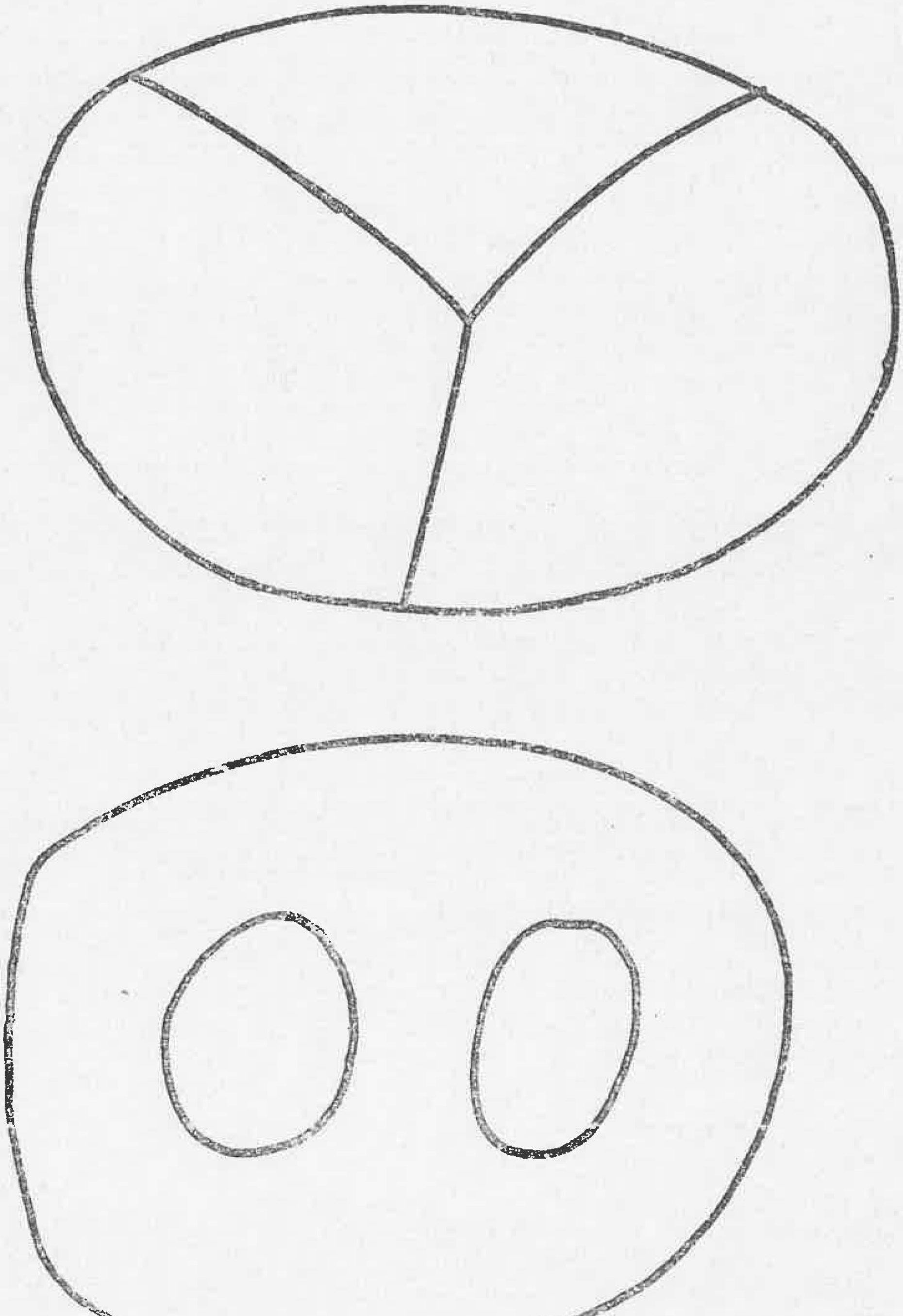


Fig 7

$T_{emn,em1}$ with $(a) \equiv 10^{a}$

Case	m \ n	0	1	2	3	4	5	6	7	8	9
A	0	0.0	-0.38102(-1)	0.0	-0.14592(-2)	0.0	-0.81066(-4)	0.0	-0.52644(-5)	0.0	-0.37314(-6)
A	1	0.0	-0.21332(-1)	0.0	-0.66705(-3)	0.0	-0.35157(-4)	0.0	-0.22281(-5)	0.0	-0.15572(-6)
B	0	0.0	0.10667(-1)	0.0	0.40848(-3)	0.0	0.22694(-4)	0.0	0.14737(-5)	0.0	0.10445(-6)
B	1	0.0	0.13676(-1)	0.0	0.42763(-3)	0.0	0.22538(-4)	0.0	0.14283(-5)	0.0	0.99823(-7)
C	0	0.0	-0.25173(-1)	0.37539(-1)	-0.13775(-1)	0.17243(-1)	-0.63722(-2)	0.80437(-2)	-0.30713(-2)	0.39574	-0.15512(-2)
C	1	0.0	-0.84157(-2)	0.23471(-1)	-0.34796(-2)	0.98554(-2)	-0.14829(-2)	0.44423(-2)	-0.68906(-3)	0.21459	-0.34110(-3)
D	0	0.0	-0.54703(-1)	0.0	-0.83798(-2)	0.0	-0.18622(-2)	0.0	-0.48372(-3)	0.0	-0.13714(-3)
D	1	0.0	-0.47939(-1)	0.0	-0.59962(-2)	0.0	-0.12641(-2)	0.0	-0.32045(-3)	0.0	-0.89582(-4)
E	0	0.0	-0.10174(±0)	0.0	-0.10824(-2)	0.0	-0.21236(-2)	0.0	-0.52953(-3)	0.0	-0.14731(-3)
E	1	0.0	-0.68765(-1)	0.0	-0.66754(-2)	0.0	-0.13014(-2)	0.0	-0.32217(-3)	0.0	-0.89156(-4)