The Scattering of Electric Waves by a Dielectric Sphere. By A. E. H. Love.

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1. The problem of the diffraction of light by small particles of spherical shape has been considered, from the point of view of the electro-magnetic theory of light, by Lord Rayleigh, in the *Philosophical Magazine* for August, 1881. An approximate solution of the problem is there given, which depends on the two suppositions: (1) that the dielectric constants for the material of the sphere and the external medium are very nearly equal, (2) that the radius of the sphere is very small compared with the wave-length of the incident light. The most important results are that, if terms of the lowest order that occur are alone retained, the waves scattered in any direction perpendicular to the direction of propagation of the incident light are completely polarized, that this result still holds good if the difference of dielectric constants is not small, and that when a second approximation is made the direction in which the scattered wave is most nearly polarized makes a slightly obtuse angle with the direction of propagation of the incident waves.

It seemed to me that it might be not without interest to work out a complete solution of the problem when the incident waves are plane, the sphere is of any size, and the difference of the dielectric constants of the internal and external medium is any given number. We should expect such a complete solution to verify exactly Lord Rayleigh's approximate result for very small spheres when terms of the lowest order only are retained, and to point to the same kind of conclusions for somewhat larger spheres when a second approximation is worked out.

2. The analysis requisite for such a complete solution has been developed by Prof. Lamb in a series of papers in the *Proceedings* (Vols. XIII. and XV.), and it will only be necessary here to make a brief statement of the equations and the types of solution employed.

In a dielectric medium with dielectric constant K and magnetic permeability μ, the equations satisfied by the electric force (X, Y, Z)

and the magnetic force (α, β, γ) are

$$\left.\begin{aligned} \frac{K}{c}\frac{\partial}{\partial t}(X, Y, Z) &= \text{curl}\,(\alpha, \beta, \gamma), \\ -\frac{\mu}{c}\frac{\partial}{\partial t}(\alpha, \beta, \gamma) &= \text{curl}\,(X, Y, Z), \end{aligned}\right\} \qquad (1)$$

wherein by the curl of a vector (u, v, w) is meant the vector whose resolved parts parallel to the axes are

$$\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}, \quad \frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}, \quad \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y},$$

and the constant c is the velocity of light in free ether.

The above equations, when (X, Y, Z) and (α, β, γ) are proportional to the same simple harmonic function of the time $e^{\iota\kappa_1 ct}$, show that the resolved parts of these vectors are related solutions of the same system of equations

$$\left.\begin{aligned} (\nabla^2+\kappa^2)\,u = 0, \quad (\nabla^2+\kappa^2)\,v = 0, \quad (\nabla^2+\kappa^2)\,w = 0, \\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0, \end{aligned}\right\} \qquad (2)$$

where $$\kappa^2 = \kappa_1^2 K\mu.$$

The solutions of these equations which involve spherical surface harmonics, and are finite at the origin, fall into two types. The first type is given by

$$(u, v, w) = \psi_n(\kappa r)\left(y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}, \quad z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z}, \quad x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)\omega_n, \quad (3)$$

where ω_n is a spherical solid harmonic of positive degree n, and

$$\psi_n(\eta) = (-)^n\, 1.3.5 \ldots (2n+1)\left(\frac{1}{\eta}\frac{d}{d\eta}\right)^n \frac{\sin\eta}{\eta}; \qquad (4)$$

the numerical coefficient is so chosen that

$$\text{Lt}_{\eta=0}\,\psi_n(\eta) = 1.$$

Since curl (u, v, w) satisfies the same equations as (u, v, w), we obtain the second type. In fact, the resolved part parallel to x of curl (u, v, w), where u, v, w are as above, is

$$-(n+1)\,\psi_{n-1}(\kappa r)\frac{\partial\omega_n}{\partial x}+\frac{n\kappa^2 r^{2n+3}}{(2n+1)(2n+3)}\psi_{n+1}(\kappa r)\frac{\partial}{\partial x}\frac{\omega_n}{r^{2n+1}}, \qquad (5)$$

and the resolved parts parallel to y and z can be put down by writing $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$ for $\frac{\partial}{\partial x}$.

The functions $\psi_n(\eta)$ satisfy certain sequence equations which can be written

$$\eta \frac{d}{d\eta}\psi_n(\eta) = -\frac{\eta^2}{2n+3}\psi_{n+1}(\eta) = (2n+1)\{\psi_{n-1}(\eta)-\psi_n(\eta)\}. \quad (6)$$

The two types of solutions which correspond to waves propagated outwards are of the same form as those which correspond to disturbances which are finite at the origin, and are obtained from them by writing everywhere $E_n(\kappa r)$ for $\psi_n(\kappa r)$, where

$$E_n(\eta) = (-)^n 1.3.5 \ldots (2n+1)\left(\frac{1}{\eta}\frac{d}{d\eta}\right)^n \frac{e^{-\iota\eta}}{\eta}, \quad (7)$$

and the functions $E_n(\eta)$ satisfy the same sequence equations as the functions $\psi_n(\eta)$.

With a view to the satisfaction of conditions at a spherical surface, we require the radial components of the above vectors. The vector given by the solutions of the first type is purely transverse. The radial component of the vector given by the solutions of the second type written as above (5) is easily seen to be

$$-\frac{1}{r}n(n+1)\psi_n(\kappa r)\omega_n. \quad (8)$$

If, then, we add to the components given by (5) and the similar forms the quantities such as

$$xr^{-2}n(n+1)\psi_n(\kappa r)\omega_n,$$

we shall have the parts contributed to these components by the transverse components of the vector, and after a little reduction we obtain forms of which the type is

$$\left\{\frac{n}{2n+1}\psi_n(\kappa r)-\psi_{n-1}(\kappa r)\right\}\left\{(n+1)\frac{\partial\omega_n}{\partial x}+nr^{2n+1}\frac{\partial}{\partial x}\frac{\omega_n}{r^{2n+1}}\right\}. \quad (9)$$

This is the part of the x-component of curl (u, v, w) contributed by the transverse components of curl (u, v, w).

The corresponding expressions in the case of waves propagated outwards are of the same forms with $E_n(\kappa r)$ in place of $\psi_n(\kappa r)$.

3. We shall now suppose that the space outside a sphere of radius R is occupied by free ether for which K and μ are both unity,

and that there is incident upon the sphere a train of plane polarized waves propagated in the negative direction of the axis of z. In these incident waves we shall suppose the electric force is parallel to y, and is given by

$$Y = Ae^{\iota\kappa(ct+z)}, \tag{10}$$

and that the magnetic force is parallel to x, and is given by

$$\alpha = Ae^{\iota\kappa(ct+z)}; \tag{11}$$

these forms are compatible with the equations (1) when K and μ are each unity. With a view to the satisfaction of the boundary conditions, we wish to have these forces expressed by means of solutions of the first type and solutions of the second type which are finite at the origin. The first step is to expand $e^{\iota\kappa z}$ in terms of surface harmonics. This expansion is known to be

$$e^{\iota\kappa z} = 1 + \sum_{n=1}^{n=\infty} \frac{(\iota\kappa r)^n}{1.3.5\ldots(2n-1)} \psi_n(\kappa r) P_n\left(\frac{z}{r}\right). \tag{12}$$

Now consider electric and magnetic forces given by

$$\left.\begin{aligned} X = 0, \quad Y = Ae^{\iota\kappa ct} \sum_0^\infty \psi_n(\kappa r) V_n, \quad Z = 0, \\ \alpha = Ae^{\iota\kappa ct} \sum_0^\infty \psi_n(\kappa r) V_n, \quad \beta = 0, \quad \gamma = 0, \end{aligned}\right\} \tag{13}$$

where V_n is a spherical solid harmonic of degree n, $V_0 = 1$, and, for $n > 1$,

$$V_n = \frac{(\iota\kappa r)^n}{1.3.5\ldots(2n-1)} P_n\left(\frac{z}{r}\right). \tag{14}$$

We can arrange the expressions for these forces as sums of solutions of the first type and solutions of the second type. Let the solutions of the first type that occur in the expression of the electric force be formed with solid harmonics of which the one of degree n is ϕ_n, and let the solutions of the first type that occur in the expression of the magnetic force be formed with solid harmonics of which the one of degree n is χ_n. Then the complete expressions for X, Y, Z are of the forms

$$\begin{aligned} X = \Sigma\psi_n(\kappa r)\left(y\frac{\partial\phi_n}{\partial z} - z\frac{\partial\phi_n}{\partial y}\right) \\ + \Sigma\frac{1}{\iota\kappa}\left[-(n+1)\psi_{n-1}(\kappa r)\frac{\partial\chi_n}{\partial x} + \frac{n\kappa^2 r^{2n+3}}{(2n+1)(2n+3)}\psi_{n+1}(\kappa r)\frac{\partial}{\partial x}\frac{\chi_n}{r^{2n+1}}\right], \end{aligned} \tag{15}$$

where the summation refers to the different orders of harmonics. In like manner the complete expressions for α, β, γ are of the forms

$$\alpha = \Sigma\psi_n(\kappa r)\left(y\frac{\partial\chi_n}{\partial z} - z\frac{\partial\chi_n}{\partial y}\right)$$
$$-\Sigma\frac{1}{\iota\kappa}\left[-(n+1)\psi_{n-1}(\kappa r)\frac{\partial\phi_n}{\partial x} + \frac{n\kappa^2 r^{2n+3}}{(2n+1)(2n+3)}\psi_{n+1}(\kappa r)\frac{\partial}{\partial x}\frac{\phi_n}{r^{2n+1}}\right]. \tag{16}$$

The normal components of (X, Y, Z) and (α, β, γ) at a sphere $r = R$ are respectively

$$\Sigma - \frac{n(n+1)}{\iota\kappa R}\psi_n(\kappa R)\chi_n \quad \text{and} \quad \Sigma\frac{n(n+1)}{\iota\kappa R}\psi_n(\kappa R)\phi_n,$$

in which ϕ_n and χ_n are to have their values for $r = R$. Now we know* that there cannot be two different expressions in terms of solutions of the first and second type which yield the same radial components of both electric and magnetic force at a spherical surface, and are both finite within the surface; we therefore obtain the result that, when $r = R$,

$$\left.\begin{aligned} \Sigma - \frac{n(n+1)}{\iota\kappa r}\psi_n(\kappa r)\chi_n &= A e^{\iota\kappa ct}\sum_0^\infty \psi_n(\kappa r)\frac{y}{r}V_n, \\ \Sigma \quad \frac{n(n+1)}{\iota\kappa r}\psi_n(\kappa r)\phi_n &= A e^{\iota\kappa ct}\sum_0^\infty \psi_n(\kappa r)\frac{x}{r}V_n, \end{aligned}\right\} \tag{17}$$

and, by equating surface harmonics of the same order at $r = R$, we find

$$\left.\begin{aligned} \phi^n &= \frac{A\iota\kappa e^{\iota\kappa ct}}{n(n+1)\psi_n(\kappa R)}\left[\frac{R^2}{2n+3}\psi_{n+1}(\kappa R)\frac{\partial V_{n+1}}{\partial x} - \frac{r^{2n+1}}{2n-1}\psi_{n-1}(\kappa R)\frac{\partial}{\partial x}\left(\frac{V_{n-1}}{r^{2n-1}}\right)\right], \\ \chi_n &= -\frac{A\iota\kappa e^{\iota\kappa ct}}{n(n+1)\psi_n(\kappa R)}\left[\frac{R^2}{2n+3}\psi_{n+1}(\kappa R)\frac{\partial V_{n+1}}{\partial y} - \frac{r^{2n+1}}{2n-1}\psi_{n-1}(\kappa R)\frac{\partial}{\partial y}\left(\frac{V_{n-1}}{r^{2n-1}}\right)\right]. \end{aligned}\right\} \tag{18}$$

4. The expression in the required form of the electric and magnetic forces in the incident wave has just been effected. We shall now suppose that the forces in the scattered wave are expressed by similar forms with ϕ'_n and χ'_n in place of ϕ_n and χ_n, and $E_n(\kappa r)$ instead

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* Lamb, *Phil. Trans.*, Part 2, 1883, p. 533.

of $\psi_n(\kappa r)$. We know that this is the complete expression for a system of waves propagated outwards.

The material of the obstructing sphere will be taken to be of dielectric constant K and magnetic permeability μ. We shall suppose the disturbance within the sphere to be derived from solid harmonics ϕ_n'' and χ_n''. The disturbance being proportional to $e^{\iota\kappa ct}$, the components of the forces now satisfy such equations as

$$(\nabla^2+\kappa'^2)\chi = 0,$$

where

$$\kappa'^2 = \kappa^2 K\mu, \tag{19}$$

and we accordingly take, as the type of X, Y, Z,

$$X = \Sigma\psi_n(\kappa' r)\left(y\frac{\partial\phi_n''}{\partial z} - z\frac{\partial\phi_n''}{\partial y}\right)$$
$$+\frac{1}{\iota\kappa K}\Sigma\left[-(n+1)\psi_{n-1}(\kappa' r)\frac{\partial\chi_n''}{\partial x} + \frac{n\kappa'^2 r^{2n+3}}{(2n+1)(2n+3)}\psi_{n+1}(\kappa' r)\frac{\partial}{\partial x}\left(\frac{\chi_n''}{r^{2n+1}}\right)\right], \tag{20}$$

and, as the type of α, β, γ,

$$\alpha = \Sigma\psi_n(\kappa' r)\left(y\frac{\partial\chi_n''}{\partial z} - z\frac{\partial\chi_n''}{\partial y}\right)$$
$$-\frac{1}{\iota\kappa\mu}\Sigma\left[-(n+1)\psi_{n-1}(\kappa' r)\frac{\partial\phi_n''}{\partial x} + \frac{n\kappa'^2 r^{2n+3}}{(2n+1)(2n+3)}\psi_{n+1}(\kappa' r)\frac{\partial}{\partial x}\left(\frac{\phi_n''}{r^{2n+1}}\right)\right]. \tag{21}$$

Then utilizing the expressions (9) for the contributions of such solutions as we have obtained to the tangential components of the forces at a sphere of radius r, we see that, when $r = R$, the condition of continuity of the tangential components of the electric force gives the two equations

$$\left.\begin{aligned} \psi_n(\kappa R)\,\phi_n + E_n(\kappa R)\,\phi_n' &= \psi_n(\kappa' R)\,\phi_n'', \\ \left\{\frac{n}{2n+1}\psi_n(\kappa R) - \psi_{n-1}(\kappa R)\right\}\chi_n + \left\{\frac{n}{2n+1}E_n(\kappa R) - E_{n-1}(\kappa R)\right\}\chi_n' & \\ = \frac{1}{K}\left\{\frac{n}{2n+1}\psi_n(\kappa' R) - \psi_{n-1}(\kappa' R)\right\}\chi_n'', & \end{aligned}\right\} \tag{22}$$

and the condition of continuity of the tangential components of the magnetic force gives two like equations obtained from the above by interchanging χ and ϕ, and writing $1/\mu$ for $1/K$.

We observe that ϕ_n' and ϕ_n'' are determined in terms of ϕ_n, and χ_n' and χ_n'' in terms of χ_n. We eliminate ϕ_n'' and χ_n'' and obtain

$$\phi_n'\left[\frac{n}{2n+1}E_n(\kappa R)-E_{n-1}(\kappa R)+\frac{1}{\mu}\left\{\frac{\psi_{n-1}(\kappa' R)}{\psi_n(\kappa' R)}-\frac{n}{2n+1}\right\}E_n(\kappa R)\right]$$

$$=\phi_n\left[\psi_{n-1}(\kappa R)-\frac{n}{2n+1}\psi_n(\kappa R)-\frac{1}{\mu}\left\{\frac{\psi_{n-1}(\kappa' R)}{\psi_n(\kappa' R)}-\frac{n}{2n+1}\right\}\psi_n(\kappa R)\right], \tag{23}$$

and a like equation, having $1/\mathrm{K}$ in place of $1/\mu$, connects χ_n' with χ_n.

This completes the analytical solution of the problem.

5. To interpret the results we shall assume $\mu = 1$ and investigate a first approximation and a second approximation when κR is small. In this case $\kappa' R$ also is small, and we may, for a first approximation, replace $\psi_n(\kappa R)$ and $\psi_n(\kappa' R)$ by unity for all values of n.

Introducing this value for the ψ functions into equation (23), and putting $\mu = 1$, we see that ϕ_n' vanishes for all values of n. To express χ_n' we observe that, when κR is small, the approximate value of $E_n(\kappa R)$ is

$$(2n+1)\frac{\{1.3.5\ldots(2n-1)\}^2}{(\kappa R)^{2n+1}}e^{-\iota\kappa R}, \tag{24}$$

and we thus find, as the first approximation to χ_n',

$$\chi_n' = \frac{\dfrac{n+1}{2n+1}\left(1-\dfrac{1}{\mathrm{K}}\right)}{\dfrac{1}{2n+1}\left(n+\dfrac{n+1}{\mathrm{K}}\right)}\,\frac{(\kappa R)^{2n+1}e^{\iota\kappa R}\chi_n}{\{1.3.5\ldots(2n-1)\}^2(2n+1)}. \tag{25}$$

Now, referring to (14), and introducing the value of $P_n(z/r)$, we find

$$\left.\begin{aligned} V_0 &= 1, & V_2 &= -\tfrac{1}{6}\kappa^2(2z^2-x^2-y^2), \\ V_1 &= \iota\kappa z, & V_3 &= -\frac{\iota\kappa^3}{30}\{2z^3-3z(x^2+y^2)\}. \end{aligned}\right\} \tag{26}$$

We have therefore, for a first approximation,

$$\phi_1 = \frac{A\iota\kappa e^{\iota\kappa ct}}{2}x, \quad \chi_1 = -\frac{A\iota\kappa e^{\iota\kappa ct}}{2}y,$$

and the most important terms in the expressions for the electric and magnetic forces in the scattered wave are given by

$$\left.\begin{aligned} X, Y, Z &= \frac{-2}{\iota\kappa} E_0(\kappa r)\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\chi_1' + \frac{1}{\iota\kappa}\frac{\kappa^2 r^5}{15} E_2(\kappa r)\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\left(\frac{\chi_1'}{r^3}\right), \\ (a, \beta, \gamma) &= E_1(\kappa r)\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y},\ z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z},\ x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\chi_1'. \end{aligned}\right\} \tag{27}$$

To see what these become at a great distance from the sphere we observe that, when κr is great, the approximate value of $E_n(\kappa r)$ is

$$(\iota)^n 1.3 \ldots (2n+1)\frac{e^{-\iota\kappa r}}{(\kappa r)^{n+1}}, \tag{28}$$

and we hence find, as approximate forms for the electric and magnetic forces in the scattered wave,

$$\left.\begin{aligned} (X, Y, Z) &= \frac{K-1}{K+2}\frac{\kappa^3 R^3}{r} A e^{\iota\kappa(ct-r+R)}\left(-\frac{xy}{r^2}, \frac{x^2+z^2}{r^2}, -\frac{yz}{r^2}\right), \\ (a, \beta, \gamma) &= \frac{K-1}{K+2}\frac{\kappa^2 R^3}{r} A e^{\iota\kappa(ct-r+R)}\left(-\frac{z}{r}, \quad 0, \quad \frac{x}{r}\right). \end{aligned}\right\} \tag{29}$$

This agrees with Lord Rayleigh's approximate results, and shows that the disturbance in the scattered wave vanishes (to the order adopted) along the line $x = 0$, $z = 0$. It follows that, for very small particles, the scattered wave corresponding to an unpolarized train of incident waves should be polarized in a direction at right angles to the direction of propagation of the incident waves, and the plane of polarization should be parallel to this direction of propagation.

It is perhaps worthy of note that, to the lowest order the effect in the scattered wave at a distance, given by (29), is the same as that of a simple Hertzian oscillator with its axis parallel to the direction of the electric force in the incident waves.

6. For a second approximation the most important consideration is that ϕ_1' no longer vanishes. When terms of order $\kappa^2 R^2$ are retained

$$\psi_n(\kappa R) = 1 - \frac{\kappa^2 R^2}{2(2n+3)}, \tag{30}$$

and the coefficient of ϕ_n on the right-hand side of (23) becomes, for $n = 1$ and $\mu = 1$,

$$1 - \frac{\kappa^2 R^2}{6} - \tfrac{1}{3}\left(1 - \frac{\kappa^2 R^2}{10}\right) - \left(1 - \frac{\kappa^2 R^2}{10}\right)\left\{\frac{1 - \frac{\kappa'^2 R^2}{6}}{1 - \frac{\kappa'^2 R^2}{10}} - \tfrac{1}{3}\right\},$$

which is $-\frac{1}{15}(\kappa^2 R^2 - \kappa'^2 R^2)$, or $\frac{K-1}{15}\kappa^2 R^2$,

since $\kappa'^2 = \kappa^2 K$.

Hence, to order $\kappa^5 R^5$, we have

$$\phi_1' = \frac{K-1}{45}\kappa^5 R^5 e^{\iota\kappa R}\phi_1, \tag{31}$$

and the additional terms thus introduced into $X, Y, Z, \alpha, \beta, \gamma$ at a great distance are

$$\left.\begin{aligned} &\text{for } (X, Y, Z) &&= -\frac{K-1}{30}\frac{\kappa^4 R^5}{r} A e^{\iota\kappa(ct-r+R)}\left(0, \frac{z}{r}, -\frac{y}{r}\right), \\ &\text{and for } (\alpha, \beta, \gamma) &&= \frac{K-1}{30}\frac{\kappa^4 R^5}{r} A e^{\iota\kappa(ct-r+R)}\left(\frac{y^2+z^2}{r^2}, -\frac{xy}{r^2}, -\frac{xz}{r^2}\right). \end{aligned}\right\} \tag{32}$$

The introduction of these terms shows that the forces in the scattered wave vanish more nearly in a direction given by

$$x = 0, \quad \frac{z}{r} = \frac{K+2}{30}\kappa^2 R^2. \tag{33}$$

This is in accordance with the result of observation that for somewhat larger particles the scattered wave is more nearly polarized in a direction inclined at a slightly obtuse angle to the direction of propagation of the incident wave. It agrees also in general character, though not numerically, with Lord Rayleigh's second approximation.

7. The second approximation to the form of ϕ_1', which vanishes to a first approximation, has introduced into the expressions for the forces terms of the order $\kappa^4 R^5$. It appears to be desirable to obtain expressions for the forces which shall be complete approximations of this order, that is shall contain all terms of order not exceeding $\kappa^4 R^5$. To do this we shall require to carry the equation connecting χ_1' and χ_1 to a higher order than before, and we shall also need a second approximation to the value of χ_1; further, we shall have to investigate whether any of the higher harmonics $\phi_2 \ldots \chi_2 \ldots$ yield any terms of order $\kappa^4 R^5$, and to evaluate any such terms if they can occur.

The complete expression for χ_1 is

$$\chi_1 = -\frac{A\iota\kappa e^{\iota\kappa ct}}{2\psi_1(\kappa R)}\left[\frac{R^2}{5}\psi_2(\kappa R)\frac{\kappa^2}{3}y + \psi_0(\kappa R)\,y\right]$$

$$= -\frac{A\iota\kappa e^{\iota\kappa ct}}{2}\,y\left(1+\frac{2}{45}\kappa^2R^2\right) \tag{34}$$

as far as terms of order κ^2R^2.

Again, we have, by (23),

$$\chi_1'\left[\tfrac{1}{3}E_1(\kappa R) - E_0(\kappa R) + \frac{1}{K}\left(\tfrac{2}{3} - \frac{\kappa'^2R^2}{15}\right)E_1(\kappa R)\right]$$

$$= \chi_1\left[1 - \frac{\kappa^2R^2}{6} - \tfrac{1}{3}\left(1 - \frac{\kappa^2R^2}{10}\right) - \frac{1}{K}\left(\tfrac{2}{3} - \frac{\kappa'^2R^2}{15}\right)\left(1 - \frac{\kappa^2R^2}{10}\right)\right], \tag{35}$$

where we have put in the second approximation to the ψ functions, and we have to put

$$E_0(\kappa R) = \frac{e^{-\iota\kappa R}}{\kappa R}, \quad E_1(\kappa R) = 3\,\frac{1+\iota\kappa R}{\kappa^3R^3}\,e^{-\iota\kappa R}. \tag{36}$$

We find, to the order κ^5R^5,

$$\chi_1' = \frac{K-1}{K+2}\,\tfrac{2}{3}\kappa^3R^3e^{\iota\kappa R}\left[1 - \iota\kappa R - \kappa^2R^2\left\{\tfrac{11}{10} - \frac{6K}{5(K+2)}\right\}\right]\chi_1. \tag{37}$$

Hence so far as χ_1 is concerned the expressions obtained in the first approximations to the forces are to be multiplied by

$$1 - \iota\kappa R - \kappa^2R^2\left\{\tfrac{11}{10} - \frac{6K}{5(K+2)}\right\} \; 1+\tfrac{2}{15}\kappa^2R^2.$$

Again, we find that the terms of lowest order in χ_2' are given by the approximate forms

$$\left.\begin{aligned} \chi_2 &= \frac{A\kappa^2e^{\iota\kappa ct}}{6}\,yz, \\ \chi_2' &= \frac{3(K-1)}{2K+3}\,\frac{\kappa^5R^5e^{\iota\kappa R}}{45}\,\chi_2, \end{aligned}\right\} \tag{38}$$

and we have contributions to the expressions for X, Y, Z and (α, β, γ) in the scattered wave of the forms

$$\left.\begin{aligned} (X, Y, Z) &= \frac{-3}{\iota\kappa}E_1(\kappa r)\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\chi_2' + \frac{2}{\iota\kappa}\,\frac{\kappa^2r^7}{35}E_3(\kappa r)\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\left(\frac{\chi_2'}{r^5}\right), \\ (\alpha, \beta, \gamma) &= E_2(\kappa r)\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y},\; z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z},\; x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\chi_2'. \end{aligned}\right\} \tag{39}$$

The terms thus contributed to the expressions for (X, Y, Z) at a great distance are

$$\frac{K-1}{2K+3}\,\frac{\kappa^4 R^5}{r}\,Ae^{\iota\kappa(ct-r+R)}\left[\tfrac{1}{3}\frac{yz}{r^2}\left(\frac{x}{r},\ \frac{y}{r},\ \frac{z}{r}\right)-\tfrac{1}{6}\left(0,\ \frac{z}{r},\ \frac{y}{r}\right)\right]. \tag{40}$$

In like manner the terms thus contributed to the expressions for (α, β, γ) at a great distance are

$$\frac{K-1}{2K+3}\,\frac{\kappa^4 R^5}{r}\,Ae^{\iota\kappa(ct-r+R)}\,\tfrac{1}{6}\left(0,\ \frac{xy}{r^2},\ \frac{-xz}{r^2}\right). \tag{41}$$

It appears on inspection of the ratios ϕ_2'/ϕ_2 and χ_3'/χ_3 that we have now exhausted all the terms of order $\kappa^4 R^5$.

Thus the complete expressions, as far as terms of order $\kappa^4 R^5$, for the forces in the scattered wave at a great distance are

$$\begin{aligned}(X, Y, Z) = {} & \frac{K-1}{K+2}Ae^{\iota\kappa(ct-r+R)}\left(-\frac{xy}{r^2},\ \frac{x^2+z^2}{r^2},\ -\frac{yz}{r^2}\right)\frac{\kappa^2R^3}{r} \\ & \qquad \times\left\{1-\iota\kappa R-\kappa^2R^2\left(\tfrac{19}{18}-\tfrac{6}{5}\frac{K}{K+2}\right)\right\} \\ & +(K-1)\,Ae^{\iota\kappa(ct-r+R)}\frac{\kappa^4R^5}{r}\,\tfrac{1}{30}\left(0,\ -\frac{z}{r},\ \frac{y}{r}\right) \\ & +\frac{K-1}{2K+3}\,Ae^{\iota\kappa(ct-r+R)}\frac{\kappa^4R^5}{r}\,\tfrac{1}{6}\left(0,\ -\frac{z}{r},\ -\frac{y}{r}\right) \\ & +\frac{K-1}{2K+3}\,Ae^{\iota\kappa(ct-r+R)}\frac{\kappa^4R^5}{r}\,\tfrac{1}{3}\frac{yz}{r^2}\left(\frac{x}{r},\ \frac{y}{r},\ \frac{z}{r}\right),\end{aligned} \tag{42}$$

and

$$\begin{aligned}(\alpha, \beta, \gamma) = {} & \frac{K-1}{K+2}Ae^{\iota\kappa(ct-r+R)}\left(-\frac{z}{r},\ 0,\ \frac{x}{r}\right)\frac{\kappa^2R^3}{r} \\ & \qquad \times\left\{1-\iota\kappa R-\kappa^2R^2\left(\tfrac{19}{18}-\tfrac{6}{5}\frac{K}{K+2}\right)\right\} \\ & +(K-1)\,Ae^{\iota\kappa(ct-r+R)}\frac{\kappa^4R^5}{r}\,\tfrac{1}{30}\left(\frac{y^2+z^2}{r^2},\ -\frac{xy}{r^2},\ -\frac{xz}{r^2}\right) \\ & +\frac{K-1}{2K+3}\,Ae^{\iota\kappa(ct-r+R)}\frac{\kappa^4R^5}{r}\,\tfrac{1}{6}\left(0,\ \frac{xy}{r^2},\ -\frac{xz}{r^2}\right).\end{aligned} \tag{43}$$

The results are in agreement with observation inasmuch as they show that there is no direction in which the forces in the scattered wave completely vanish, and that the intensity of the residual disturbance in the direction in which it most nearly vanishes varies inversely as the eighth power of the wave-length. It is noteworthy that, if terms of order $\kappa^3 R^4$ are retained and terms of order $\kappa^4 R^5$ neglected, the scattered wave is precisely as given by the first approximation (29), except that the exponential factor becomes $e^{\iota\kappa(ct-r)}$.

8. A very similar analysis applies to the problem when the material of the sphere is regarded as conducting. If σ is the specific resistance, the equations that hold within the sphere become

$$\left.\begin{aligned}\left(\frac{K}{c}\frac{\partial}{\partial t}+\frac{4\pi c}{\sigma}\right)(X,\ Y,\ Z) &= \text{curl}\,(\alpha,\ \beta,\ \gamma),\\ -\frac{\mu}{c}\frac{\partial}{\partial t}(\alpha,\ \beta,\ \gamma) &= \text{curl}\,(X,\ Y,\ Z),\end{aligned}\right\}\tag{44}$$

and it follows that, when all the forces are proportional to $e^{\iota\kappa ct}$, each of them satisfies an equation of the form

$$(\nabla^2+\kappa'^2)\,u=0,$$

where
$$\kappa'^2=\kappa^2 K\mu-\frac{4\pi\mu\,\iota\kappa c}{\sigma}=\kappa^2\mu\left\{K-\frac{4\pi\iota c}{\kappa\sigma}\right\}.\tag{45}$$

The two circuital relations hold as before, and the forces at points within the sphere are connected by the equations

$$\left.\begin{aligned}(X,\ Y,\ Z) &= \frac{1}{\iota\kappa K+4\pi c/\sigma}\,\text{curl}\ (\alpha,\ \beta,\ \gamma),\\ (\alpha,\ \beta,\ \gamma) &= \frac{-1}{\iota\kappa\mu}\,\text{curl}\ (X,\ Y,\ Z).\end{aligned}\right\}\tag{46}$$

The forms of the expressions for the forces at internal points are easily written down in terms of two systems of solid harmonics ϕ''_n and χ''_n, and the forms at external points are the same as before. In the boundary conditions we have to put

$$\frac{1}{K-4\pi\iota c/\kappa\sigma}\ \text{in place of}\ \frac{1}{K},$$

wherever $1/K$ occurs explicitly.

The result of the first approximation still holds good, provided $|\kappa' R|$ is small when κR is small. With frequencies of orders of magnitude corresponding to visible light K is the square of the refractive index, and the quantity $4\pi c/\kappa\sigma$ is small compared with K for badly conducting materials, and thus a slight degree of conductivity does not appreciably affect the result.* For a good conductor, however, K may be omitted, and

$$\kappa'^2 = -\frac{4\pi\mu\iota\kappa c}{\sigma}.$$

I find that for frequency and wave-length corresponding to the D lines of the solar spectrum, and with σ equal to the specific resistance of copper,

$$\frac{\kappa'}{\kappa} = \sqrt{(-\iota)}\, 45 \text{ nearly.}$$

To make $|\kappa' R|$ small we should require $45\kappa R$ to be small, or, since $\kappa = 2\pi/\lambda$, where λ is the wave-length, this would require R to be about $\lambda/300$ to make $|\kappa' R|$ about 1/10. Such a value of R is so near to molecular dimensions that a continuous analysis could not be applied to the problem. On the other hand, if we could imagine the resistance to be very much less than it is for the best conductors, we might make an approximation on the supposition that $\kappa' R$ is great while κR is small.

In writing out this approximation we put $\mu = 1$, and

$$\kappa' = \frac{1-\iota}{\sqrt{2}} \sqrt{\left(\frac{4\pi c}{\kappa\sigma}\right)} = (1-\iota)\,\vartheta \text{ say;}$$

then

$$\psi_0(\kappa' R) = \frac{\sin \kappa' R}{\kappa' R} = \frac{e^{\vartheta R} e^{\iota\vartheta R}}{2\iota\kappa' R}$$

approximately when ϑR is great. Also we have

$$\psi_1(\kappa' R) = -\frac{3}{\kappa'^3 R^3}(\kappa' R \cos \kappa' R - \sin \kappa' R)$$

$$= -\frac{3}{\kappa'^3 R^3}\tfrac{1}{2}(\kappa' R + \iota)\, e^{\vartheta R} e^{\iota\vartheta R}$$

approximately when ϑR is great. This gives

$$\frac{\psi_0(\kappa' R)}{\psi_1(\kappa' R)} = \frac{\kappa'^2 R^2}{3}\,\frac{1}{1-\iota\kappa' R} = \frac{\iota}{3}\kappa' R \qquad (47)$$

approximately when $\kappa' R$ is great.

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* *Cf.* G. W. Walker, *Quarterly Journal of Pure and Applied Mathematics*, Vol. xxx., 1898, p. 217.

The equation connecting χ_1' and χ_1 now becomes

$$\chi_1' \left[\tfrac{1}{3} E_1(\kappa R) - E_0(\kappa R) + \frac{\kappa^2}{\kappa'^2}\left(\frac{\iota}{3}\kappa' R\right) E_1(\kappa R)\right] = \chi_1 \left[\tfrac{2}{3} - \frac{\kappa^2}{\kappa'^2}\left(\frac{\iota}{3}\kappa' R\right)\right],$$

where we are to put

$$E_0(\kappa R) = \frac{e^{-\iota\kappa R}}{\kappa R}, \quad E_1(\kappa R) = \frac{3}{\kappa^3 R^3} e^{-\iota\kappa R}.$$

Thus the most important part of χ_1' is given by

$$\chi_1' = \tfrac{2}{3}\kappa^3 R^3 e^{\iota\kappa R}\chi_1. \tag{48}$$

Again, the equation connecting ϕ_1' and ϕ_1 becomes

$$\phi_1' = -\tfrac{1}{3}\kappa^3 R^3 e^{\iota\kappa R}\phi_1, \tag{49}$$

when only the most important terms are retained.

With these approximations the expressions for the forces at a distance become

$$(X, Y, Z) = \frac{\kappa^2 R^3}{r} A e^{\iota\kappa(ct-r+R)} \left(-\frac{xy}{r^2}, \frac{x^2+z^2}{r^2}, -\frac{yz}{r^2}\right)$$
$$+ \tfrac{1}{2}\frac{\kappa^2 R^3}{r} A e^{\iota\kappa(ct-r+R)} \left(0, \frac{z}{r}, -\frac{y}{r}\right), \tag{50}$$

and
$$(\alpha, \beta, \gamma) = \frac{\kappa^2 R^3}{r} A e^{\iota\kappa(ct-r+R)} \left(-\frac{z}{r}, 0, \frac{x}{r}\right)$$
$$- \tfrac{1}{2}\frac{\kappa^2 R^3}{r} A e^{\iota\kappa(ct-r+R)} \left(\frac{y^2+z^2}{r^2}, -\frac{xy}{r^2}, -\frac{xz}{r^2}\right). \tag{51}$$

These expressions verify Prof. J. J. Thomson's result* that, for a perfect conductor, the forces in the scattered wave vanish in the direction $x = 0$, $z/r = -\frac{1}{2}$, *i.e.*, in a direction making an angle $\frac{1}{3}\pi$ with the direction of propagation of the incident waves. This result could have been more simply obtained by neglecting the disturbance inside the sphere and taking the electric force at the boundary to be purely radial. We have shown above that there is no reason for thinking that this investigation could have any application to the problem of the scattering of light by small particles, though it might conceivably represent something that could be observed for Hertzian waves a metre long and metallic spheres of a few millimetres radius.

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* *Recent Researches*, p. 448.

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